Logic, Proofs and Counterexamples[1]

This is an informal discussion of some basic points of logic in Mathematics. There is nothing obsure or mysterious about this discussion; with the exception of "proof by induction", the logic involved is just plain old "common sense" of the sort you use in every-day life (I hope!). However experience has shown that many students forget about every-day logic in a mathematical context, so it may be useful to discuss these matters explicitly.

A. Assertions, Converses and Contrapositives

Consider the following examples.

1. **Assertion**: If I live in Texas, then I live in the United States.
   Converse: If I live in the United States, then I live in Texas.
   Contrapositive: If I don't live in the United States, then I don't live in Texas.

2. **Assertion**: If the sea is boiling hot, then pigs have wings.[2]
   Converse: If pigs have wings, then the sea is boiling hot.
   Contrapositive: If pigs don't have wings, then the sea is not boiling hot.

3. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a map.
   Assertion: If T is linear, then $T(0) = 0$.
   Converse: If $T(0) = 0$, then T is linear.
   Contrapositive: If $T(0) \neq 0$, then T is not linear.

The pattern should be clear. An assertion is just a statement of the form "If P is true, then Q is true" (the assertion itself may be either true or false, as in the above examples). The converse is the assertion "If Q is true, then P is true". The contrapositive is the assertion "If Q is false, then P is false".

Note the following.

4. Even if the original assertion is true, the converse may well be false. This happens in example 1, for instance – living in the United States does not imply living in Texas. This also happens in example 3 – the original assertion is true, but the converse is false. The point I wish to make is that the logic involved is identical. No sane individual would ever say "living in Texas implies living in the United States; THEREFORE, living in the United States implies living in Texas". By the same token, although assertion 3 is true, the converse statement is false; the fact that $T(0) = 0$ does NOT imply that T is linear (more on example 3 below).

5. On the other hand, it IS true that any assertion is equivalent to its contrapositive. In other words, if the assertion is true, then its contrapositive is true. Sound complicated? It isn't – just look at example 1. Again, I emphasize that the truth or falsity of the original assertion is not the relevant point here – the point is that the assertion and its contrapositive are logically equivalent. For instance, suppose in place of example 1

[1] This handout is adapted (with permission) from a handout written by Steve Mitchell (Professor, University of Washington).

[2] Trivia quiz: can you name the literary source of this example? Can you recite the relevant verse? A partial answer is on the last page. The author, incidentally, was an amateur Mathematician.

that a visitor from the planet Glop informs us that “If I live in Nithgor, then I live in Irxyanithon”. Ignorant as we are of the geography of the planet Glop, we have no way to confirm or deny this assertion. However, we DO know that it is logically equivalent to the assertion “If I don’t live in Irxyanithon, then I don’t live in Nithgor”. Either both are true, or both are false.

B. Proofs and Counterexamples

If you want to prove the assertion “P implies Q”, then you must show that in ALL cases where P is true, Q is also true; but if you want to show that the assertion “P implies Q” is false, all you have to do is produce ONE example (called a counterexample) in which P is true but Q is false. Once again, I want to emphasize that there is nothing arcane nor mysterious here – it is just plain old common-sense logic. Consider for example the following assertions.

6. Every country in South America is smaller than France.
   This is false – Brazil is much bigger than France, and so is a “counterexample” to assertion 6. Of course, there are other counterexamples too – e.g., Argentina – but that’s irrelevant. All we have to do to DISPROVE the assertion is to find ONE counterexample.

7. Every country in South America is bigger than Austria.
   This assertion happens to be true. But you certainly couldn’t prove it by saying “Brazil is bigger than Austria”. To prove it, you’d have to get out a *reliable* atlas and check EVERY country in South America (the smallest, French Guyana, is only slightly bigger than Austria).

The two preceding examples probably seem utterly trivial and obvious. However, many students allow themselves to be tripped up by the exact same simple logic in mathematical problems....

8. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map; if $T(0) = 0$, then T is linear.
   This is false. To disprove it, you only need ONE counterexample. For instance, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x_1, x_2) = (x_1^2, x_2)$. Then $T(0) = 0$, yet T is not linear. You do need to PROVE that this really is a counterexample, however. That is, you need to show that $T(0) = 0$ (obvious), and you need to show that T is not linear.

9. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map; if T is linear, then $T(0) = 0$.
   This is true, and was proved in class. For now, I just want to comment on some WRONG “proofs”. For instance, suppose you say “well, it’s true for the linear map $T(\mathbf{v}) = 3\mathbf{v}$, and therefore is true for all linear T”. That would be absurd – it would be analogous to “proving” assertion 7 by merely noting that Brazil is bigger than Austria. The point is that you must show that EVERY linear map maps the origin to the origin. Furthermore, in contrast to example 7, there are infinitely many linear maps, so it is not possible, even in principle, to prove assertion 9 by a case by case check. Nor is it enough to prove it for $n = 2$; you have to prove it for arbitrary n and arbitrary T – more sophisticated reasoning is required.

C. Proof by Contradiction

In its simplest form, proving an assertion "by contradiction" just means proving the contrapositive (remember that the assertion and the contrapositive are equivalent). In other words, to prove that P implies Q, one supposes that Q is false and shows that this implies that P was false to begin with. In practice, however, it often works like the following. Suppose we want to show that P implies Q. We assume that (1) P is true and that (2) Q is false, and see what can be deduced from this. If, at any stage of our deductions, we arrive at a contradiction (e.g, $1 = 0$), then our second assumption "Q is false" is false. That is, if P is true, then it's impossible to have Q be false, so "P implies Q". (Equivalently, one may view (1) as false, so that "Q false implies P false", which is the contrapositive).

Example: The square root of 2 is irrational.
Here P is the assertion "x is the square root of 2" (i.e., $x^2 = 2$) and Q is the assertion "x is irrational". A sketch of the proof is as follows. Suppose $x^2 = 2$ (P is true) and that x is rational (Q is false). Then we may write $x = a/b$ where a and b are integers which are not both even (so one could be even, and the other odd). Then $2 = x^2 = a^2/b^2$, which implies a^2 is even. Writing a as a product of primes and then analysing a^2 shows that a is even; thus $a^2 = 4c$ for some integer c. However $b^2 = a^2/2 = 2c$, so b^2 is even. Writing b as a product of primes and then analysing b^2 shows that b is even. In particular, we have shown that BOTH a and b are even integers, which contradicts an earlier sentence. Hence, x is irrational. ■

D. Proof by Induction

Consider the problem of finding a formula for the sum of the first n positive integers. Call this number s_n. That is, $s_1 = 1$, $s_2 = 1 + 2 = 3$, $s_3 = 1 + 2 + 3 = 6$, etc. Let's say that after experimenting for a while you find that for $n < 12$ you have $s_n = n(n + 1)/2$. You suspect that your formula is true for all positive integers n, but how can you prove it??! Of course, with a modern computer you can easily check the formula up to a million or so – but this still doesn't prove it for all values of n. One method to prove it is the following (called mathematical induction). Suppose you have a sequence of assertions, one for each positive integer n. In the present example, the nth assertion is that $s_n = n(n + 1)/2$. Suppose you can show that:

(i) the assertion is true for some suitable initial value(s) of n;

(ii) if the assertion is true for n, then it is true for $n + 1$.

Then the assertion is indeed true for all positive integers n larger than your choice of initial n. I won't try to justify the principle of induction here, but note that it makes sense: condition (i) gets the ball rolling, and condition (ii) keeps it rolling. If the assertion is true for $n = 1$, then by (ii) it is true for $n = 2$, so by (ii) again, it is true for $n = 3$, and so on ad infinitum.

Returning to our example, note that (i) is trivial, namely $s_1 = 1(1+1)/2$. We now suppose the assertion is true for n; i.e., we assume $s_n = n(n + 1)/2$. We must show the assertion is true for $n + 1$; i.e., we must show that $s_{n+1} = (n + 1)(n + 2)/2$. By definition, we have $s_{n+1} = s_n + (n + 1)$, and using the inductive hypothesis, we have $s_n = n(n + 1)/2$, therefore $s_{n+1} = n(n + 1)/2 + (n + 1)$. The latter simplifies to $(n + 1)(n + 2)/2$, which completes the proof. ■

E. Exercises (for your interest and amusement only – not to be turned in)

1. State the converse and contrapositive of each of the following assertions (don't worry about whether or not they are true):
   - if the Cowboys ever win the Superbowl, then it will snow in Texas in July;
   - if a function is differentiable, then it is continuous;
   - every snake is a reptile (you might want to rephrase this one as an "if/then" statement).

2. Assertions: all of the insurance agents are athletes; none of the professors are insurance agents. Write a true sentence which involves only professors and athletes (not insurance agents).

3. Why is the sketch proof given above that $\sqrt{2}$ is irrational only a sketch proof? That is, why is it not formal enough to be a full-blown proof?

4. There is another way to prove that the sum of the first n positive integers is $n(n+1)/2$ that a 10-year-old child could understand; can you find it?

5. Prove the following assertions by mathematical induction:
   - there are exactly $n-1$ multiplications performed in multiplying n generic numbers (meaning that you may assume none are zero nor equal to one) (note: assume n is a positive integer);
   - there are exactly $n-1$ additions performed in adding n nonzero numbers, where n is a positive integer.

6. **Assertion**: if there are n billiard balls on a billiard table, then all n balls have the same color.
   'Proof' (by induction). If $n = 1$, then the assertion is true. Suppose the result is true for n. We will prove it is true for $n + 1$. Suppose there are $n + 1$ balls on the table. Remove one ball. Then there are n balls left on the table. So, by the inductive hypothesis, those n balls have the same color. Return the removed ball back to the table, and remove another ball. Again, by the inductive hypothesis, the remaining n balls have the same color. Hence all $n + 1$ balls have the same color. ■

   As you know, the assertion is, of course, false! So there has to be something wrong with this proof. What's wrong with it?!

The time has come to talk of many things: of shoes and ships and sealing wax, of cabbages and kings, and why the sea is boiling hot and whether pigs have wings. Lewis Carroll