This sheet accompanies the homework website for the course. Any page numbers or sections refer to the book by Dummit & Foote, 3rd Ed.

H1. Prove that a group G is abelian iff $(ab)^2 = a^2b^2$ for all a, $b \in G$.

H2. Let n be a positive integer; use the subgroup criterion to prove that the subset of upper-triangular matrices in $GL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$.

H3. For $H \leq G$ and $g \in G$, prove that $gH = Hg$ iff $g \in N_G(H)$.

H4. For $\sigma = (1\ 3\ 5)(2\ 4)$ and $\tau = (1\ 5)(2\ 3)$, write each of the maps σ, τ, σ^2, $\sigma\tau$, $\tau\sigma$, $\tau^2\sigma$ as a product of transpositions and determine which of these 6 maps is even and which is odd (hint: recall ϵ is a homomorphism, so as to avoid doing a lot of multiplication).

H5. Let S be any ring and n a positive integer; prove that the subset of upper-triangular matrices in $M(n, S)$ is a subring of $M(n, S)$.

H6. For a subset X of a ring R, prove that if $rx \in X$ for all $r \in R$ and $x \in X$, then X is closed under multiplication; deduce that a left ideal is a sub***group*** I of R such that $rx \in I$ for all $r \in R$ and $x \in I$.

H7. Without using factor rings, prove that $p\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$ for all prime numbers p (hint: use Euclidean Algorithm).

H8. Prove that $\mathbb{C}[x, y]/\langle y \rangle \cong \mathbb{C}[x]$ (hint: define a surjective ring homomorphism $\mathbb{C}[x, y] \to \mathbb{C}[x]$ and use FIT).

H9. Use the previous question to prove that $\langle y \rangle$ is a prime ideal of $\mathbb{C}[x, y]$.

H10. If P is a completely prime ideal in a ring R, prove that $R \setminus P$ is closed under multiplication.

H11. For $D = \mathbb{Z} \setminus 2\mathbb{Z}$, is $D^{-1}\mathbb{Z}$ defined? If so, what is it; if not, why not?

H12. (optional) For $R = \mathbb{C}[X]/\langle X(X-1)\rangle$ and
$D = \{f + \langle X(X-1)\rangle \in R : f \in \mathbb{C}[X] \text{ has nonzero constant term}\}$,
show that $D^{-1}R \cong \mathbb{C}$ (hint: write $x = X + \langle X(X-1)\rangle$ and show $x = 0$ in $D^{-1}R$).

H13. (optional) For $R = \mathbb{C}[X]/\langle X^2(X-1)\rangle$ and
$D = \{f + \langle X^2(X-1)\rangle \in R : f \in \mathbb{C}[X] \text{ has nonzero constant term}\}$, describe $D^{-1}R$.

H14. Let $I = \langle x, y\rangle$ and $J = \langle x+y, x-y\rangle$ in the ring $F[x, y, z]$, where F is a field in which $2 \neq 0$; show that $I = J$. (Hint: if $f \in$ an ideal K, then $\langle f\rangle \subseteq K$ – why?)

H15. For a ring R and left R-module M, prove that $r0 = 0$, for all $r \in R$, where $0 \in M$.

H16. For a ring R and left R-module M and $m \in M$, prove that Rm is a submodule of M.

H17. For a ring R, prove that the left R-submodules of R are the left ideals of R.

H18. For a field F and $R = M(2, F)$ and $M = F^2$, prove that if $0 \neq \begin{bmatrix} a \\ b \end{bmatrix} \in M$, then $M = R \begin{bmatrix} a \\ b \end{bmatrix}$.

H19. For a nonzero ring R with unity and a unital left R-module M, let J be any ideal of R that is contained in the annihilator of M in R (see previous question in book); define $f : R/J \times M \to M$ by $f : (r + J, m) \mapsto rm$ for all $r \in R$ and $m \in M$. Prove that f makes M into a left R/J-module (hence, if R/J is a field, then M is a vector space over R/J).

H20. For a ring R, left R-module M and fixed $m \in M$, define $\phi : R \to M$ by $\phi(r) = rm$ for all $r \in R$; prove that ϕ is an R-module homomorphism.

H21. For a ring R and left ideal I of R, prove that I is a maximal left ideal of R iff R/I is a simple left R-module (hint: 4th isom thm).

H22. For a ring R and left R-modules M_1 and M_2 and submodules $N_1 \subseteq M_1$ and $N_2 \subseteq M_2$, prove that $\dfrac{M_1 \oplus M_2}{N_1 \oplus N_2} \cong \dfrac{M_1}{N_1} \oplus \dfrac{M_2}{N_2}$ (hint: map $M_1 \oplus M_2 \to \frac{M_1}{N_1} \oplus \frac{M_2}{N_2}$, find the kernel and use FIT).

H23. Let $R = \mathbb{Z}$ and let $M = \mathbb{Z}^n$, where $n \geq 1$. Prove that any $n+1$ elements of M are $\mathbb{Z}$-linearly dependent; i.e., for any $x_1, \ldots, x_{n+1} \in M$, there exist $z_1, \ldots, z_{n+1} \in \mathbb{Z}$, not all zero, such that $\sum_{i=1}^{n+1} z_i x_i = 0$ (hint: view $M \subseteq \mathbb{Q}^n$ = vector space over $\mathbb{Q}$ of dimension n, and work inside $\mathbb{Q}^n$). Consequently, we define the *rank* of any $\mathbb{Z}$-module M to be the maximal number of $\mathbb{Z}$-linearly independent elements of M.

H24. What are the elementary divisors of the group $\mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_{35}$; what are its invariant factors? Do the same for $\mathbb{Z}_{26} \times \mathbb{Z}_{42} \times \mathbb{Z}_{49} \times \mathbb{Z}_{200} \times \mathbb{Z}_{1000}$. (Hint: use pg 164.)

H25. Let p be a prime number and $n \in \mathbb{N} \setminus \{0\}$.

(a) Prove that any cyclic group of order p^n contains an element of order p.

(b) Prove that a finite abelian group that is not cyclic contains a subgroup that is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

H26. Using the definition of solvable, show that S_2, S_3 and S_4 are solvable.

H27. Let $G = S_3$.

(a) Find $[G, G]$.

(b) Show that G is not nilpotent by finding G^i for all i.