The FINAL TEST will be in PKH 302 on Tues May 9, 5:30-8:00 pm

- See above for date, time and location of the Final Test. It is worth 38% of your course grade.
- As stated in the course information sheet, **any student who does not obtain a positive score on the Final Test will not pass this class.**
- The material to be covered on the test is the same as that covered on the homework and in lectures; see http://www.uta.edu/math/vancliff/T/S17/hwk5331.html .)
- Until Thurs May 4, my office hours are Tues & Thurs 4:40-5:20 pm in PKH 462, and the GTA's office hours are Wed 1:30-2:30 pm & Fri 11 am-noon, both in PKH 423. See the homework website for my office hours after Thurs May 4.
- When I write a test, I look over the lecture notes, previous tests and homework, and use them to model about 85% of the test problems (and most of them are fair game). You should expect the length of the test to be double the length of Test 3. The ratio of material covered after Test 3 to that covered before Test 3 is approximately 1:4.
- You should know the definitions of the concepts (and examples thereof) that arise frequently in class; the theorems etc that are used frequently or which have a name; how to apply the theorems to examples and at least one method for doing each homework question. Learn any formulae taught in class. Skimming over short proofs (at most 6 or so lines) from lecture could also be helpful. You should be ready to present examples like those occurring in the homework and in lectures, and be able to show that a subset of a group is a subgroup and a subset of a ring is a subring or a (left/right) ideal. Moreover, any question in the homework is game for the test.
- A good way to review is to go over the homework problems you have not already done, & make sure you understand all the homework well by 48 hours prior to the test. You should also go over Tests 1-3, which can be accessed from the homework page (see above). Consider adopting one or more of the study techniques linked at our website.
- It is your responsibility to be on time.
- Try to keep your eyes on your own work during the test.
- BRING YOUR MYMAV ID CARD WITH YOU. All other nontest items will have to be put by the side walls prior to the start of the test; this includes calculators, cell phones and all electronic devices (including apple watches), which need to be switched off & in with your belongings by the wall – noncompliance will result in a score of 0 on the test.
- If you wish to leave the room during the test, you should ask permission first & turn in your test to me. Only in exceptional circumstances will I let you continue the test should you return. (So it is better to be 3 minutes late to the test, rather than ask to go to the bathroom during the test.)
- I would like to take this opportunity to remind y'all to complete the student feedback survey for all your classes by 11 pm Fri May 5; see your mymav email for the link. Thank you!

Groups typically arise in the world in the form of functions acting on a set or on a manifold, etc. E.g., matrices acting on a vector space, or functions that rotate and reflect a square. For this reason, groups are key players in chemistry in the study of electrons in an atom, and in the subject of robotics (as well as in many other fields).

We started the semester with definitions and examples and studied group actions, which is a topic that abstractifies the idea of functions acting on a set. In particular, the symmetric group of permutations of a set Ω yields a group action on Ω.

Subgroups were then considered, and we proved the Subgroup Criterion to help us establish when a subset of a group is a subgroup. We studied special subgroups, such as the center of a group, the centralizer of a subset of a group, the normalizer of a subset of a group, and the stabilizer of a subset of a set that is being acted on by a group.

Homomorphisms were considered briefly, and mainly (at this stage) for the goal of helping us in our discussion of cyclic groups. E.g., all finite cyclic groups of order n are isomorphic to each other and hence isomorphic to $\mathbb{Z}/n\mathbb{Z}$, and all infinite cyclic groups are isomorphic to $\mathbb{Z}$.

We briefly considered the idea of a subset of a group generating the group. This is very convenient for cyclic groups, which are generated by a single element, and also for some other groups, such as dihedral groups.

We then returned to homomorphisms and began studying their fibers. We establised that the set of fibers of a homomorphism is in fact a group, with multiplication defined appropriately. This group is called the quotient group (or the factor group, depending on the book). We met the First Isomorphism Theorem (FIT) in this context: if $\phi : G \to G'$ is a group homomorphism and if $K = \ker(\phi)$, then $G/K \cong \phi(G)$, where the quotient group G/K is the group of fibers of ϕ. We also considered the idea of a coset of a subgroup, as the notion of coset generalizes the notion of fiber of a homomorphism (as each fiber is a coset of K). Trying to force the set of all cosets of a subgroup be a group (in our attempt to mimic our success with the fibers of a homomorphism) leads to the idea of a normal subgroup of a group G; that is, subgroups H such that $gH = Hg$ for all $g \in G$. Normal subgroups N of a group G can be "factored" out of G to produce a quotient group G/N. (If $|G| < \infty$, then $|G/N| = |G|/|N|$, and this is why the quotient group is denoted G/N.)

At this stage, we studied Lagrange's Theorem and Cauchy's Theorem, which apply to finite groups. Lagrange's Theorem tells us that the order of any subgroup of a finite group G divides the order of G. Cauchy's Theorem tells us that if G is a finite group with order a multiple of a prime number, p, then G contains an element of order p (and therefore contains at least one subgroup of order p (the cyclic group generated by any element of order p)). Although our textbook does not prove Cauchy's Theorem at this stage, we discussed a proof of it that is outlined in one of the exercises in the book (pg 96: 9).

After considering more group isomorphism theorems and the idea of a composition series, we returned to the symmetric group. We defined the sign function on permutations leading to the idea of even, and odd, permutations. This, in turn, led to the definition of the alternating group, which is the subgroup of the symmetric group given by the even permutations. Note that a single odd-length cycle is even

and a single even-length cycle is odd!! We noted that the alternating group on n symbols is a simple group, except when $n = 2$ or 4.

At this stage, we turned our attention to rings, which are abelian groups under addition that also have a second operation, called multiplication, that behaves "well" with respect to addition. We studied some examples of rings such as $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, polynomial rings, matrix rings and group rings. We proved some results concerning zero divisors and units (invertible elements). In our attempt to duplicate the homomorphism results of groups for rings, we defined a ring homomorphism and found ourselves needing an analog of normal subgroup, namely an ideal. The concept of ideal allows us to define a quotient ring. (The weaker concepts of left/right ideal do not lead to quotient rings, but will be useful to us when we study modules in Chapter 10.) We found that the group isomorphism theorems have analogs for rings.

Some special ideals are principal ideals and finitely generated ideals, the former being ideals that can be generated by only one element and the latter being ideals that have a finite generating set; all ideals of $\mathbb{Z}$ are principal. Some other special ideals are maximal ideals and prime ideals and these concepts are intimately related to fields, division rings and domains via the notion of quotient ring. Any prime ideal in $\mathbb{Z}$ is generated by a prime number or 0, and this is what led to the name of this type of ideal. Also, all maximal ideals are prime, but not conversely ($\{0\}$ in $\mathbb{Z}$ is prime but not maximal).

Sometimes, it is helpful to embed a ring or integral domain R in a larger ring Q in order to obtain inverses for some, or all, nonzero elements of R. This is akin to embedding the ring $\mathbb{Z}$ in the field $\mathbb{Q}$, and is the process of "localization". The name comes from algebraic geometry, where one associates rings to collection of points and the act of "zooming in" on one or more points corresponds to inverting elements in a ring (i.e., going from global to local).

The analog of the notion of group action in the setting of rings brought us to the notion of a module over a ring, which generalizes the idea of a vector space over a field. We "repeated" many of the concepts and results from groups and rings in this setting: submodules, homomorphisms, quotient modules and the first isomorphism theorem. The appropriate generalization of a basis of a vector space over a field brought us to the idea of a free module over a ring.

At this point, we returned to study groups and did so via group actions. For a group G and a subgroup H of G, we considered G acting on the cosets of H. This brought us to Cayley's Theorem which allows G to be represented by a subgroup of an appropriately sized symmetric group. However, considering G acting on itself via conjugation brought us to the Class Equation, which is sometimes helpful in determining properties of G, such as the property of being abelian.

Certain groups, such as p-groups, can be helpful in determining properties of groups that contain them. This approach brought us to Sylow's Theorem, and the proof of that result relied heavily on group actions.

After Sylow's Theorem, we concentrated on understanding finitely generated abelian groups. By the Fundamental Theorem of Finitely Generated Abelian Groups, we know that such a group is either a free group or is the direct product of a free group with finite cyclic groups (where the free group could be $\mathbb{Z}^0 = \{1\}$). Our approach to the proof entailed viewing the abelian group as a $\mathbb{Z}$-module and using a matrix to display the relations. Row operations and column operations allow one to simplify

the relations and write the group more simply and, in our case, helped our proof of the fundamental theorem. Associated to any abelian group are invariant factors and elementary divisors. It is advisable to know how to pass back and forth between the two (see the homework for such questions).

Our next, and final, topic was solvable groups and nilpotent groups. Abelian groups are nilpotent, and nilpotent groups are solvable. Hence, these concepts generalize the notion of abelian group. (The word "solvable" is used since such groups show up in the theory of solving polynomial equations in one variable, and are helpful in proving that a generic quintic polynomial in one variable cannot be solved by using a radical expression alone (unlike the quadratic, cubic and quartic cases).) We only skimmed the surface of this last topic, but these concepts have counterparts in the theory of Lie algebras, and so their study is a rich subject and is helpful in the classification of various types of groups.

The common theme throughout this course has been the idea of groups arising as collections of functions acting on an object. We saw this in group actions, in the proof of Sylow's theorem and in many of the other theorems, and again in the idea of a ring acting on a module. It is a theme that has allowed abstract algebra to be so useful in applications and has allowed abstract algebra to grow so fruitfully.