Math 4321

Here are proofs of 2 results in Section 51 that we might not have proved in class today.

Thm (a) If $f \in F[x]$ is irred, then all zeros of f in $\bar{F}$ have the same mult. (b) See lecture notes.

Proof (a) Let $\alpha, \beta \in \bar{F}$ be zeros of f. So $\psi_{\alpha,\beta} : F(\alpha) \xrightarrow{\cong} F(\beta)$ and extn thm $\Rightarrow$ can extend $\psi_{\alpha,\beta}$ to an isom $\tau : \bar{F} \to \bar{F}$. Hence, τ induces an isom $\sigma : \bar{F}[x] \to \bar{F}[x]$ where $\sigma\left(\sum_{i=0}^{j} a_i x^i\right) = \sum_{i=0}^{j} \tau(a_i) x^i$. Also, $\sigma(f) = f$ since $\tau(a) = \psi_{\alpha,\beta}(a) = a \ \forall\, a \in F$ and $f \in F[x]$. However, $\sigma((x-\alpha)^m) = (x-\beta)^m$, so the mult of β in $f \geq$ mult of α in f. Since $\sigma((x-\beta)^r) = (x-\alpha)^r$, we obtain the reverse inequality. $\Rightarrow$ equality.

((a) $\Rightarrow$ (b) as discussed in class). ■

Lemma Suppose $g \in F[x]$, F = field, g monic. If $g = f^m$ in $\bar{F}[x]$ for some monic $f \in \bar{F}[x]$ and if $m \cdot 1 \neq 0$ in F, then $f \in F[x]$.

Proof See pg 439-40 in book. ■

Thm Every field of charac 0 is perfect.

Proof Suppose $F \subseteq E$ fields, $\text{char}(F) = 0$, $[E:F] < \infty$.

Let $\alpha \in E$ and $f = \text{irr}(\alpha, F)$. In $\overline{F}[x]$, $f = \prod_{i=1}^{n} (x - \alpha_i)^m$

where the α_i are the distinct zeros of f. Write $\alpha_1 = \alpha$.

So, $f = \left(\prod_{i=1}^{n} (x - \alpha_i) \right)^m$ and, since $m \cdot 1 \neq 0$ in F, previous lemma $\Rightarrow$

$\prod_{i=1}^{n} (x - \alpha_i) \in F[x]$. This is a contradiction if $m \geq 2$. $\Rightarrow m = 1$

$\Rightarrow$ α sep over F.

α arbitrary $\in E \Rightarrow$ every element of E is sep over F

$\Rightarrow E$ is a sep extn of F. ■