Math 4321

Here are proofs of 3 results in Section 31 that we might not have proved in class today.

Thm If E = alg extn of a field F, then $\exists\ \alpha_1, \ldots, \alpha_n \in E$ $(n < \infty)$ such that $E = F(\alpha_1, \ldots, \alpha_n)$ iff $[E:F] < \infty$.

Proof $\Rightarrow$ Suppose $E = F(\alpha_1, \ldots, \alpha_n)$ for some $\alpha_1, \ldots, \alpha_n \in E$ that are alg over F. α_j alg over $F \Rightarrow \alpha_j$ alg over $F(\alpha_1, \ldots, \alpha_{j-1})$ $\forall j$.

$\Rightarrow F \leq F(\alpha_1) \leq F(\alpha_1, \alpha_2) \leq F(\alpha_1, \alpha_2, \alpha_3) \leq \cdots \leq F(\alpha_1, \ldots, \alpha_n) = E$.

alg simple extn $\Rightarrow$ dim $< \infty$ by Thm 30.23

Tower Thm $\Rightarrow [E:F] < \infty$.

$\Leftarrow$ If $[E:F] = 1$, then done (ie $E = F(\alpha), \alpha = 1$). If $E \neq F$, choose $\alpha_1 \in E \setminus F$. If $E = F(\alpha_1)$, then done. If not, choose $\alpha_2 \in E \setminus F(\alpha_1)$, ... Continue. Since $[E:F] < \infty$, this process terminates after finitely many steps. ■

Thm If E = extn of a field F, then the set $\overline{F}_E = \{\alpha \in E : \alpha \text{ alg over } F\}$ is a subfield of E.

Proof Since $0 \in \overline{F}_E$, $\overline{F}_E$ nonempty. Let $\alpha, \beta \in \overline{F}_E$. By Thm 30.23 and Tower Thm, $[F(\alpha,\beta):F] < \infty \Rightarrow F(\alpha,\beta)$ = alg extn of F. $\Rightarrow F(\alpha,\beta) \subseteq \overline{F}_E$. Hence $\alpha - \beta$, $\alpha\beta \in \overline{F}_E$; and $\alpha^{-1}, \beta^{-1} \in \overline{F}_E$ if $\alpha \neq 0, \beta \neq 0$. Since α, β arbitrary, it follows that $\overline{F}_E$ is a subring of E in which every nonzero element is invertible; thus, $\overline{F}_E$ is a subfield of E. ■

Cor An algebraically closed field has no proper alg extns.

Proof Let E = alg extn of an alg closed field F. We will prove $E=F$.

Let $\alpha \in E$. E = alg extn $\Rightarrow \alpha$ alg over $F \Rightarrow \alpha$ = zero of some $f \in F[x] \setminus F$.

F alg closed $\Rightarrow$ may assume $\deg(f)=1$ (by using a thm from class).

So $f = ax-b$ where $a \in F^\times$, $b \in F$. Thus, $\alpha = b/a \in F$.

α arbitrary $\Rightarrow E \subseteq F$. Since $F \subseteq E$, have $E = F$. ■