Math 4321

3/24/15

Here is the proof of the result in Section 27 that we did not prove in class today.

However, first recall that a monic poly is a poly with leading coeff $= 1$.

Thm Let F, E be fields with $F \leq E$, and let $\alpha \in E$ be alg over F. There is an irred $p \in F[x]^{\times}$ such that $p(\alpha) = 0$ and p has least degree ≥ 1 in $F[x]$ among all polys in $F[x]$ that vanish on α. and p can be uniquely chosen to be monic. If $f \in F[x]^{\times}$ and $f(\alpha) = 0$, then $p \mid f$ (i.e., $f \in \langle p \rangle$).

Proof $\phi_\alpha : F[x] \to E$ is eval hom. $\ker(\phi_\alpha)$ = ideal of $F[x]$ and ideals of $F[x]$ are principal. $\Rightarrow \ker(\phi_\alpha) = \langle f \rangle$ for some $f \in F[x]$. $\ker(\phi_\alpha) \neq \langle 0 \rangle$ since α is alg over F, so $f \neq 0$. Definition of $\phi_\alpha \Rightarrow \langle f \rangle = \ker(\phi_\alpha) = \{$all polys in $F[x]$ that vanish on $\alpha\}$.

Thus, f is a poly of smallest degree ≥ 1 that vanishes on α.

Write $f = \sum_{i=0}^{n} a_i x^i$, $a_i \in F$ $\forall i$, $a_n \neq 0$ and let $p = a_n^{-1} f$. So $p \in \langle f \rangle$. and so $\langle p \rangle \subseteq \langle f \rangle$. Also, $f = a_n p \Rightarrow f \in \langle p \rangle \Rightarrow \langle f \rangle \subseteq \langle p \rangle$.

Hence, $\langle p \rangle = \langle f \rangle = \ker(\phi_\alpha)$.

It follows that p is the unique monic poly of least deg that vanishes on α. It remains to prove that p is irred.

Suppose $p = gh$, where $g, h \in F[x]$, $\deg(g) < \deg(p)$, $\deg(h) < \deg(p)$. (*)

Since $p(\alpha) = 0$, $g(\alpha)h(\alpha) = 0$ in E (= field) $\Rightarrow g(\alpha) = 0$ or $h(\alpha) = 0$.

$\Rightarrow g \in \langle p \rangle$ or $h \in \langle p \rangle \Rightarrow p \mid g$ or $p \mid h$, which contradicts (*).

Hence, p is irred.