Math 3330. End of Sec 3.4 — 3/17/15

Defn We say an $n \times n$ matrix B is **similar** to an $n \times n$ matrix A if there exists an invertible $n \times n$ matrix S such that

$$B = S^{-1}AS.$$

i.e. 2 matrices are similar iff they represent the same lin transf (but wrt different bases.).

Eg/ Are $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ similar?

Sol Want $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (if it exists) such that $B = S^{-1}AS$ (where $a, b, c, d \in \mathbb{R}$).

ie want $SB = AS$. $SB = \begin{bmatrix} 5a & -b \\ 5c & -d \end{bmatrix}$, $AS = \begin{bmatrix} a+2c & b+2d \\ 4a+3c & 4b+3d \end{bmatrix}$

↖ set equal and solve ↗

$S = \begin{bmatrix} a & b \\ 2a & -b \end{bmatrix}$ $\forall\, a, b \in \mathbb{R}$. ⇐

Since want S to be invertible, try $a = 1 = b \Rightarrow S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$. is invertible and solves $S^{-1}AS = B \Rightarrow A$ and B are similar matrices. (Note: there are many matrices S that solve $S^{-1}AS = B$.).

Eg. B sim to $A \Rightarrow B^t$ sim to A^t $\forall\, t \in \mathbb{R}$. since

$$B = S^{-1}AS \Rightarrow B^t = \underbrace{(S^{-1}AS)(S^{-1}AS)\cdots(S^{-1}AS)}_{t \text{ factors}} = S^{-1}A^tS.$$

Eg/ Suppose B is sim to A and C is sim to B, is C sim to A?

Sol $\left.\begin{matrix} B = S^{-1}AS \\ C = P^{-1}BP \end{matrix}\right\} \Rightarrow C = P^{-1}BP = P^{-1}(S^{-1}AS)P = (SP)^{-1}A(SP).$

$\Rightarrow C$ sim to A.