This sheet accompanies the homework website for the course.

As usual, $\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$, and V denotes any vector space over $\mathbb{F}$ and V' denotes the dual of V. Any page numbers or sections refer to Axler's book, 3rd Ed.

H1. $\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$. Find the dual basis of $(\mathbb{R}^3)'$ relative to $\mathcal{B}$.

H2. For each of the following symmetric matrices, give the corresponding quadratic form.

(a) $A = \begin{bmatrix}1 & 2 & 3\\2 & 4 & 5\\3 & 5 & 6\end{bmatrix}$ (b) $A = \begin{bmatrix}0 & 1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$ (c) $A = \begin{bmatrix}1 & 0 & -1 & 0\\0 & 3 & 0 & 2\\-1 & 0 & 0 & 0\\0 & 2 & 0 & 2\end{bmatrix}$.

H3. For each of the following quadratic forms Q, find a symmetric matrix A such that $Q(v) = v^T A v$ for all $v \in V$.

(a) $Q(x_1, x_2) = 6x_1^2 - 7x_1x_2 + 8x_2^2$.

(b) $Q(x_1, x_2) = x_1x_2$.

(c) $Q(x_1, x_2, x_3) = x_1x_2$.

(d) $Q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 6x_1x_3 + 7x_2x_3$.

H4. Find the minimal polynomial (in factored form) of each of the following matrices:

$$\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}, \quad \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}, \quad \begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}, \quad \begin{bmatrix}2 & 1 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}, \quad \begin{bmatrix}2 & 1 & 0\\0 & 2 & 1\\0 & 0 & 2\end{bmatrix}.$$

In each case, describe the minimal polynomial in terms of the appearance of the matrix. What is the pattern?

H5. Repeat the previous question for the following matrices:

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}.$$

H6. Repeat the previous question for the following matrices: $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

H7. Suppose $\mathbb{F} = \mathbb{C}$. Let $\dim(V) < \infty$ and let $T \in \mathcal{L}(V)$ be diagonalizable. Prove that if $U \subseteq V$ is an invariant subspace, then $T|_U$ is diagonalizable. (Hint: use question 12 in Section 8C.)

H8. Suppose $\mathbb{F} = \mathbb{C}$. Let $\dim(V) < \infty$ and let $T,\ S \in \mathcal{L}(V)$ where T and S are diagonalizable (possibly with respect to different bases). Prove that there exists a basis of V with respect to which both T and S have diagonal matrices if and only if T and S commute on V. (Hint for "if": show $E(\lambda,\ S)$ is invariant under T for all eigenvalues λ of S and use the previous question.)
(In this case, T and S are said to be *simultaneously diagonalizable*. This result is used a lot in the theory of Lie algebras.)

H9. Suppose $\mathbb{F} = \mathbb{C}$. Find the Jordan normal (canonical) form of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

and find an invertible matrix P such that $P^{-1}AP$ is the Jordan normal form of A.

H10. Suppose $\mathbb{F} = \mathbb{C}$. Find the Jordan normal (canonical) form of the matrix

$$A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$$

and find an invertible matrix P such that $P^{-1}AP$ is the Jordan normal form of A. (Hint: refer to the model example on the class handout.)

H11. Suppose $\mathbb{F} = \mathbb{C}$. Find the Jordan normal (canonical) form of the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and find an invertible matrix P such that $P^{-1}AP$ is the Jordan normal form of A. (Hint: refer to the model example on the class handout.)

H12. Matrices A and B are said to be *similar* if there exists an invertible matrix Q such that $A = Q^{-1}BQ$. Suppose $\mathbb{F} = \mathbb{C}$. Find the Jordan normal (canonical) form of the following pairs of matrices and state which are similar.

(a)
$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(b) (Here $i^2 = -1$.)
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

(c)
$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

H13. Let $T \in \mathcal{L}(V)$ and let $\mathcal{B}_1$ and $\mathcal{B}_2$ denote two bases on V. Let $A_1 = M(T,\ \mathcal{B}_1,\ \mathcal{B}_1)$ and let $A_2 = M(T,\ \mathcal{B}_1,\ \mathcal{B}_2)$.

(a) Find an example with $V = \mathbb{F}^2$ where A_1 and A_2 do <u>not</u> have the same eigenvalues.

(b) Why does (a) not contradict the fact that similar matrices have the same eigenvalues?

H14. Suppose $\mathbb{F} = \mathbb{R}$. Find the Jordan normal (canonical) form of the matrix
$$A = \begin{bmatrix} 37 & 10 & -5 & 5 \\ 10 & 22 & 10 & -10 \\ -5 & 10 & 37 & 5 \\ 5 & -10 & 5 & 37 \end{bmatrix},$$
given that its characteristic polynomial is $(x-7)(x-42)^3$, and find an <u>orthogonal</u> matrix P such that P^TAP is the Jordan normal (canonical) form of A. (Hint: P is orthogonal if P is invertible and $P^{-1} = P^T$; notice that A is symmetric (self-adjoint), and apply the Gram-Schmidt orthogonalization procedure to each eigenspace of A in turn.)

H15. Suppose $\mathbb{F} = \mathbb{C}$. Find the Jordan normal (canonical) form of the matrix
$$A = \begin{bmatrix} 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ -3 & 3 & 2 & 8 & -2 & 0 & -1 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 \end{bmatrix},$$
given that its characteristic polynomial is $(x-2)^5(x-3)^2$, and find an invertible matrix P such that $P^{-1}AP$ is the Jordan normal (canonical) form of A. (Hint: refer to the model example on the class handout.)

H16. (From January 2011 Preliminary Examination A)

Suppose $\mathbb{F} = \mathbb{R}$. Let $A = \begin{bmatrix} 7 & -6 & -3 \\ 8 & -7 & -4 \\ -4 & 4 & 3 \end{bmatrix}$. The characteristic polynomial of A is $(x-1)^3$.

(a) How many linearly independent eigenvectors does A have? Is this information alone, after knowing the characteristic polynomial, enough to determine the Jordan normal (canonical) form J of A (up to rearrangement of blocks)? Explain your answer. If this information is enough, what is J?

(b) Find a matrix P such that $P^{-1}AP$ is the Jordan normal (canonical) form of A. (Hint: refer to the model example on the class handout.)