The required text is

- Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra, Undergraduate Texts in Mathematics, by David A. Cox, John Little, Donal O'Shea, Springer, 3rd edition; denoted [CLO] in this course.

but other books that might be helpful are

- Undergraduate Algebraic Geometry, by M. Reid, Cambridge University Press
- Using Algebraic Geometry, by D. Cox, J. Little, D. O'Shea, 2nd Ed., Graduate Texts in Mathematics, Vol. 185, Springer-Verlag
- Geometry, by E. Rees, Notes on Geometry, Universitext, Springer-Verlag
- Algebraic Geometry, by R. Hartshorne, Graduate Texts in Mathematics, Vol. 52, Springer-Verlag
- Algebraic Geometry: a First Course, by J. Harris, Graduate Texts in Mathematics, Vol. 133, Springer-Verlag
- Schemes: The Language of Modern Algebraic Geometry, by D. Eisenbud and J. Harris, Wadsworth & Brooks/Cole Mathematics Series
- The Geometry of Schemes, by D. Eisenbud and J. Harris, Graduate Texts in Mathematics, Vol. 197, Springer-Verlag
- Principles of Algebraic Geometry, by P. Griffiths and J. Harris, Wiley and Sons
- Introduction to Commutative Algebra and Algebraic Geometry, by E. Kunz, Birkhaüser
- Elementary Algebraic Geometry, by K. Hulek, Student Mathematical Library, Vol. 20, AMS Publications
- Algebraic Geometry, (3 books) by K. Ueno, Translations of Mathematical Monographs, AMS Publications
- Algebraic Curves and One-Dimensional Fields, by F. Bogomolov and T. Petrov, AMS Publications
- Elementary Algebraic Geometry, K. Kendig, Graduate Texts in Mathematics, Vol. 44, Springer-Verlag.

I have not seen all these books; many were recommended to me by others.

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Not all questions need to be attempted to get a grade of "A", but the more correctly completed questions, the better the grade. You may work together on the homework, but submit solutions individually.

In the following questions, $\mathbb{C}$ denotes the complex numbers, $\mathbb{R}$ denotes the real numbers, $\mathbb{Z}$ denotes the set of all integers, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{Q}$ denotes the rational numbers and $\Bbbk$ denotes a field. Also, the symbol $\subset$ will be used to denote "contained in, or equal to,".

1. Read pages 1-4 of [CLO].

2. Do Exercise 1 on page 5 of [CLO].

3. Do Exercise 2 on page 5 of [CLO], which is.... Let $\Bbbk = \mathbb{F}_2$ denote the field with two elements in the preceding question.
   (a) Consider the polynomial $g(x, y) = x^2y + y^2x \in \Bbbk[x, y]$. Show that $g(x, y) = 0$ for every $(x, y) \in \Bbbk^2$, and explain why this does not contradict [CLO, Proposition 5].
   (b) Find a nonzero polynomial in $\Bbbk[x, y, z]$ that vanishes at every point of $\Bbbk^3$. Try to find one that uses all three variables.
   (c) Find a nonzero polynomial in $\Bbbk[x_1, \ldots, x_n]$ that vanishes at every point of $\Bbbk^n$. Can you find one that uses all n variables?

4. Do Exercise 3 on page 5 of [CLO] (requires Math 3321).

5. Do Exercise 5 on page 5 of [CLO], which is.... In the proof of [CLO, Proposition 5], we took $f \in \Bbbk[x_1, \ldots, x_n]$ and wrote it as a polynomial in x_n with coefficients in $\Bbbk[x_1, \ldots, x_{n-1}]$. To see what this looks like in a specific case, consider the polynomial
$$f(x, y, z) = x^5y^2z - x^4y^3 + y^5 + x^2z - y^3z + xy + 2x - 5z + 3.$$
   (a) Write f as a polynomial in x with coefficients in $\Bbbk[y, z]$.
   (b) Write f as a polynomial in y with coefficients in $\Bbbk[x, z]$.
   (c) Write f as a polynomial in z with coefficients in $\Bbbk[x, y]$.

6. Read pages 5-12 of [CLO].

7. Do Exercise 1 on page 12 of [CLO], which is.... Sketch the following affine varieties in $\mathbb{R}^2$.
   (a) $\mathcal{V}(x^2 + 4y^2 + 2x - 16y + 1)$ (b) $\mathcal{V}(x^2 - y^2)$ (c) $\mathcal{V}(2x + y - 1,\ 3x - y + 2)$.

8. Do Exercise 3 on page 12 of [CLO], which is.... In the plane $\mathbb{R}^2$, draw a picture to illustrate
$$\mathcal{V}(x^2 + y^2 - 4) \cap \mathcal{V}(xy - 1) = \mathcal{V}(x^2 + y^2 - 4,\ xy - 1),$$
   and determine the points of intersection. Note that this is a special case of [CLO, Lemma 2].

9. Do Exercise 4 on page 12 of [CLO], which is.... Sketch the following affine varieties in $\mathbb{R}^3$.
   (a) $\mathcal{V}(x^2 + y^2 + z^2 - 1)$ (b) $\mathcal{V}(x^2 + y^2 - 1)$ (c) $\mathcal{V}(x + 2,\ y - 1.5,\ z)$
   (d) $\mathcal{V}(xz^2 - xy)$ (hint: factor) (e) $\mathcal{V}(x^4 - zx,\ x^3 - yx)$
   (f) $\mathcal{V}(x^2 + y^2 + z^2 - 1,\ x^2 + y^2 + (z - 1)^2 - 1)$.
   In each case, does the variety have the dimension you would intuitively expect it to have?

10. Do Exercise 5 on page 12 of [CLO], which is.... Use the proof of [CLO, Lemma 2] to sketch
$$\mathcal{V}((x - 2)(x^2 - y),\ y(x^2 - y),\ (z + 1)(x^2 - y))$$
    in $\mathbb{R}^3$. Hint: this is the union of which two varieties?

11. For each of the following, show that V is an affine variety by expressing V as the zero locus of certain polynomials in $\mathbb{R}[x, y]$.
    (a) $V = \{(1, 0),\ (2, 0)\} \subset \mathbb{R}^2$.
    (b) $V = \{(1, 0),\ (2, 3)\} \subset \mathbb{R}^2$.

12. Do Exercise 6 on page 12 of [CLO], which is....
    (a) Prove that a single point $(a_1, \ldots, a_n) \in \Bbbk^n$ is an affine variety.
    (b) Prove that every finite subset of $\Bbbk^n$ is an affine variety. Hint: [CLO, Lemma 2] will be useful.

13. Do Exercise 8 on page 13 of [CLO], which is.... If a subset X of $\Bbbk^n$ is *not* an affine variety, then it can take some work to show X is not an affine variety. For example, consider the set
$$X = \{(x, x) : x \in \mathbb{R}, x \neq 1\} \subset \mathbb{R}^2,$$
which is the straight line $y = x$ with the point $(1, 1)$ removed. To show that X is not an affine variety, suppose that $X = \mathcal{V}(f_1, \ldots, f_s)$. Each f_i vanishes on X, so if we can show that each f_i also vanishes at $(1, 1)$, we will have a contradiction. Thus, here is what you have to prove: if $f \in \mathbb{R}[x, y]$ vanishes on X, then $f(1, 1) = 0$. Hint: let $g(t) = f(t, t)$, which is a polynomial in $\mathbb{R}[t]$; now apply the proof of [CLO, §1, Proposition 5].

14. Do Exercise 15 on page 14 of [CLO], which is.... In [CLO, Lemma 2], we showed that if V and W are affine varieties, then so are $V \cup W$ and $V \cap W$.
    (a) Prove that finite unions (respectively, intersections) of affine varieties are again affine varieties. Hint: use induction.
    (b) Give an example to show that an infinite union of affine varieties need not be an affine variety. Hint: consider the preceding 2 questions. (Surprisingly, an infinite intersection of affine varieties is still an affine variety; this is a consequence of the Hilbert Basis Theorem, which will be discussed later in the course.)
    (c) Give an example to show that the set-theoretic difference $V \setminus W$ of two affine varieties need not be an affine variety. Hint: consider the preceding 2 questions.
    (d) Let $V \subset \Bbbk^n$ and $W \subset \Bbbk^m$ be two affine varieties, and let
    $$V \times W = \{(x_1, \ldots, x_n,\ y_1, \ldots, y_m) \in \Bbbk^{n+m} : (x_1, \ldots, x_n) \in V,\ (y_1, \ldots, y_m) \in W\}$$
    be their cartesian product. Prove that $V \times W$ is an affine variety in $\Bbbk^{n+m}$. Hint: if V is defined by $f_1, \ldots, f_s \in \Bbbk[x_1, \ldots, x_n]$, then we can regard $f_1, \ldots, f_s$ as polynomials in $\Bbbk[x_1, \ldots, x_n, y_1, \ldots, y_m]$, and similarly for W. Show that this gives defining equations for the cartesian product.

15. Read pages 14-20 of [CLO].

16. Do Exercise 1 on page 23 of [CLO].

17. Do Exercise 2 on page 23 of [CLO].

18. Do Exercise 3 on page 23 of [CLO].

19. Do Exercise 4 on page 23 of [CLO], which is.... Consider the parametric representation $x(t) = t/(1+t),\ y(t) = 1 - 1/t^2$.
    (a) Find the equation (in terms of x and y only) of the affine variety determined by these parametric equations.
    (b) Show that these parametric equations parametrize all points of the variety found in (a) except for the point (1, 1).

20. Read pages 30-35 of [CLO].

21. Do Exercise 1 on page 36 of [CLO].

22. Do Exercise 2 on page 36 of [CLO], which is.... Let $I \subset \Bbbk[x_1, \ldots, x_n]$ be an ideal and let $f_1, \ldots, f_s \in \Bbbk[x_1, \ldots, x_n]$. Prove that $f_1, \ldots, f_s \in I$ if and only if $\langle f_1, \ldots, f_s \rangle \subset I$.

23. Do the following modification of Exercise 3 on page 36 of [CLO]. Use the previous question to prove the following equalities of ideals in $\Bbbk[x, y]$. (You may assume that $\mathbb{N} \subset \Bbbk$.)
    (a) $\langle x + y,\ x - y \rangle = \langle x,\ y \rangle$
    (b) $\langle x + xy,\ y + xy,\ x^2,\ y^2 \rangle = \langle x,\ y \rangle$.
    (c) $\langle 2x^2 + 3y^2 - 11,\ x^2 - y^2 - 3 \rangle = \langle x^2 - 4,\ y^2 - 1 \rangle$.

    This illustrates that the same ideal can have many different bases and that different bases may have different numbers of elements.

24. Do Exercise 5 on page 36 of [CLO], which is.... Show that $\mathcal{V}(x + xy,\ y + xy,\ x^2,\ y^2) = \mathcal{V}(x,\ y)$. (Hint: see the previous question.)

25. The following is a modification of Exercise 6(a)(b) on page 36 of [CLO].
    (a) Consider the ideal $I = \langle x \rangle \subset \Bbbk[x]$. As an ideal, I has a basis consisting of the one element x. Write down what this means using the definition of such a basis. However, since $\Bbbk[x]$ is a vector space over $\Bbbk$, I can also be regarded as a subspace of $\Bbbk[x]$. Write down the definition of a vector space basis for I. Prove that any vector-space basis of I is infinite. Hint: by theorems in linear algebra, if one vector-space basis of a vector space is infinite, then they all are, so it suffices to find one such basis for I that is infinite.
    (b) In linear algebra, a basis of a vector space V must span V and be linearly independent over $\Bbbk$, whereas, for an ideal, a basis is concerned only with spanning – there is no mention of any sort of independence. The reason is that once we allow polynomial coefficients, no independence is possible. To see this, consider the ideal $\langle x,\ y \rangle \subset \Bbbk[x, y]$. Show that zero can be written as an algebraic combination of x and y with nonzero polynomial coefficients.

26. Do Exercise 7 on page 37 of [CLO].

27. Do Exercise 8 on page 37 of [CLO].

28. Do Exercise 15(a)(b) on page 38 of [CLO].

29. Prove that if J is an ideal in $\Bbbk[x_1, \ldots, x_n]$, then $\sqrt{J} \subset I(\mathcal{V}(J))$ (the reverse containment is also true if $\Bbbk$ is algebraically closed, but is much harder to prove, so we will cover it in class).

30. Prove that if I is an ideal in $\Bbbk[x_1, \ldots, x_n]$, then $\mathcal{V}(\sqrt{I}) = \mathcal{V}(I)$.

31. Prove that if J is an ideal in $\Bbbk[x_1, \ldots, x_n]$, then $\mathcal{V}(I(\mathcal{V}(J))) = \mathcal{V}(J)$.

32. Let $I = \langle f_1, \ldots, f_s \rangle \subset \Bbbk[x_1, \ldots, x_n]$. Prove that $\mathcal{V}(I) = \mathcal{V}(f_1, \ldots, f_s)$.

33. Skim pages 38-45 of [CLO].

34. Do Exercise 1 on page 46 of [CLO].

35. Do Exercise 5 on page 46 of [CLO], which is.... If $f,\ g \in \Bbbk[x]$, prove that $\langle f - qg,\ g \rangle = \langle f,\ g \rangle$ for any $q \in \Bbbk[x]$.

36. Do Exercise 13 on page 81 of [CLO].

37. Do Exercise 14 on page 81 of [CLO].

38. Do Exercise 15 on page 81 of [CLO].

39. Do Exercise 18 on page 81 of [CLO], which is.... If an ideal has a basis where some of the elements can be factored, we can use the factorization to help understand the variety.
    (a) Show that if $g \in \Bbbk[x_1, \ldots, x_n]$ factors as $g = g_1 g_2$, then, for any $f \in \Bbbk[x_1, \ldots, x_n]$, $\mathcal{V}(f,\ g) = \mathcal{V}(f,\ g_1) \cup \mathcal{V}(f,\ g_2)$.
    (b) Show that, in $\mathbb{R}^3$, $\mathcal{V}(y - x^2,\ xz - y^2) = \mathcal{V}(y - x^2,\ xz - x^4)$.
    (c) Use (a) to describe and/or sketch the variety from (b).

40. The goal of this question is to prove that, for all $\alpha_1, \ldots, \alpha_n \in \Bbbk$, $\langle x_1 - \alpha_1, \ldots, x_n - \alpha_n \rangle$ is a maximal ideal of $\Bbbk[x_1, \ldots, x_n]$ (withOUT using quotient rings!).
    (a) Let $\alpha \in \Bbbk$. By the division algorithm on $\Bbbk[x]$, if $f \in \Bbbk[x]$, then $f = q(x - \alpha) + \beta$ where $q \in \Bbbk[x]$ and $\beta \in \Bbbk$; verify this.
    (b) Use (a) to prove that, for all $\alpha \in \Bbbk$, $\langle x - \alpha \rangle$ is a maximal ideal in $\Bbbk[x]$.
    (c) Let $\alpha, \beta \in \Bbbk$ and $f \in \Bbbk[x, y]$. Viewing $\Bbbk[x, y]$ as $(\Bbbk[x])[y]$, prove that $f = q_1(y - \beta) + r_1$, where $q_1 \in \Bbbk[x, y]$ and $r_1 \in \Bbbk[x]$, then prove $f = q_1(y - \beta) + q_2(x - \alpha) + r_2$, where $q_2 \in \Bbbk[x]$ and $r_2 \in \Bbbk$.
    (d) Use (c) to prove that, for all $\alpha, \beta \in \Bbbk$, $\langle x - \alpha, y - \beta \rangle$ is a maximal ideal in $\Bbbk[x, y]$.
    (e) See next page.

(e) Use (b), (d) and induction to prove that, for all $\alpha_1, \ldots, \alpha_n \in \Bbbk$, $\langle x_1 - \alpha_1, \ldots, x_n - \alpha_n \rangle$ is a maximal ideal in $\Bbbk[x_1, \ldots, x_n]$.

41. Do Exercise 14 on page 183 of [CLO], which is.... In $\Bbbk[x_1, \ldots, x_n]$, let $J = \langle x_1 x_2,\ (x_1 - x_2)x_1 \rangle$. Describe $\mathcal{V}(J)$ and show that $\sqrt{J} = \langle x_1 \rangle$.

42. Read pages 183-186 of [CLO].

43. Suppose $\Bbbk$ is algebraically closed and that I is an ideal of $R = \Bbbk[x_1, \ldots, x_n]$. Let $f \in R$ be such that $\mathcal{V}(f) \cap \mathcal{V}(I) = \emptyset$. Prove that there exists $r \in R$ such that $rf - 1 \in I$. (Hint: let $J = I + Rf$; what is $\mathcal{V}(J)$? Consider the weak nullstellensatz.)

44. Do Exercise 2 on page 203 of [CLO], which is.... Prove that every prime ideal is radical. (Assume the ring is $\Bbbk[x_1, \ldots, x_n]$ or any commutative ring with 1, whichever is comfortable for you.)

45. Do Exercise 5 on page 200 of [CLO], which is.... Express $f = x^2z - 6y^4 + 2xy^3z$ as $f = f_1(x + 3) + f_2(y - 1) + f_3(z - 2)$ for some $f_1, f_2, f_3 \in k[x, y, z]$.

46. If $f \in \Bbbk[x_1, \ldots, x_n]$, prove that $\langle f \rangle$ is prime if and only if f is irreducible.

47. Suppose that $\Bbbk$ is algebraically closed. Prove that if $h \in \Bbbk[x_1, \ldots, x_n]$ is irreducible, then so is $\mathcal{V}(h)$; prove the converse with the assumption that $\langle h \rangle$ is radical.

48. Read Proposition 13 on page 189 of [CLO].

49. Read pages 206.75-207.8 of [CLO].

50. In $\Bbbk[x, y, z]$, prove that $\langle xy,\ xz \rangle$ is radical, but not prime.

51. Do Exercise 4 on page 213 of [CLO].

52. Do Exercise 1 on page 220 of [CLO], which is.... Let V denote the twisted cubic curve in $\mathbb{R}^3$ and let $W = \mathcal{V}(v - u - u^2)$ in $\mathbb{R}^2$. Show that $\phi(x,\ y,\ z) = (xy,\ z + x^2y^2)$ defines a regular function from V to W. (Hint: consider a parametrization of V.)

53. Do Exercise 2 on page 220 of [CLO], which is.... Let $V = \mathcal{V}(y - x)$ in $\mathbb{R}^2$ and let $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ be the regular function represented by $\phi(x,\ y) = (x^2 - y,\ y^2,\ x - 3y^2)$. Prove that the image of V under ϕ is an affine variety in $\mathbb{R}^3$; find a system of equations defining this image.

54. Do Exercise 5 on page 221 of [CLO], which is.... Show that $\phi_1(x,\ y,\ z) = (2x^2 + y^2,\ z^2 - y^3 + 3xz)$ and $\phi_2(x,\ y,\ z) = (2y + xz,\ 3y^2)$ represent the same regular function from the twisted cubic curve in $\mathbb{R}^3$ to $\mathbb{R}^2$.

55. Do Exercise 6 on page 221 of [CLO], which is.... Consider the regular function $\phi : \mathbb{R}^2 \to \mathbb{R}^5$ defined by $\phi(u, v) = (u, v, u^2, uv, v^2)$.
    (a) The image of ϕ is a variety V known as an affine Veronese surface; find polynomial equations defining V.
    (b) Show that the projection $\pi : V \to \mathbb{R}^2$ defined by $\pi(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2)$ is the inverse mapping of ϕ taking $\mathbb{R}^2$ to V. What does this imply about V and $\mathbb{R}^2$?

56. Read pages 222-227 of [CLO]. However, Definition 8(iii) on pg 225 is not correct; look in other books to find what is wrong with it!

57. Do Exercise 5 on page 228 of [CLO].

58. Do Exercise 6 on page 228 of [CLO].

59. Parts (a) and (b) work you through a variant of Proposition 8 on pg 235 of [CLO]; you might want to read that proof for inspiration! Let $J \subset \Bbbk[x_1, \ldots, x_n]$ be an ideal and write $V = \mathcal{V}(J)$.
    (a) Prove that if $|V| < \infty$, then $|V| \leq \dim_{\Bbbk}\left(\frac{\Bbbk[x_1, \ldots, x_n]}{J}\right)$. (Hint:
        (i) Write $V = \{p_1, \ldots, p_m\} \subseteq \Bbbk^n$. Prove that there exist $f_i \in \Bbbk[x_1, \ldots, x_n]$ such that, for all i, $f_i(p_i) = 1$ and $f_i(p_j) = 0$ if $j \neq i$.
        (ii) Prove that $\left\{[f_i] \in \frac{\Bbbk[x_1, \ldots, x_n]}{J}\right\}_{i=1}^{m}$ is a linearly independent set and explain why (a) now follows.)
    (b) Suppose that $\Bbbk$ is an algebraically closed field and $|V| < \infty$. Prove that if $\sqrt{J} = J$ (i.e., J is radical), then $|V| = \dim_{\Bbbk}\left(\frac{\Bbbk[x_1, \ldots, x_n]}{J}\right)$. (Hint: prove that the $[f_i]$ in part (a)(ii) span $\frac{\Bbbk[x_1, \ldots, x_n]}{J}$.)
    (c) Let $\Bbbk = \mathbb{R}$ and $V = \mathcal{V}(x^2 + 1)$. Show that $|V| < \dim_{\mathbb{R}}\left(\frac{\mathbb{R}[x]}{\langle x^2 + 1\rangle}\right)$, so that equality in (b) can fail if $\Bbbk$ is not an algebraically closed field, even if $\sqrt{J} = J$.
    (d) Let $\Bbbk$ denote any field and suppose that $\dim_{\Bbbk}\left(\frac{\Bbbk[x_1, \ldots, x_n]}{J}\right) < \infty$.
        (i) Prove that $\dim_{\Bbbk}\left(\frac{\Bbbk[x_1, \ldots, x_n]}{\sqrt{J}}\right) \leq \dim_{\Bbbk}\left(\frac{\Bbbk[x_1, \ldots, x_n]}{J}\right)$.
        (ii) Prove that $|V| < \infty$ and that $|V| \leq \dim_{\Bbbk}\left(\frac{\Bbbk[x_1, \ldots, x_n]}{\sqrt{J}}\right)$.

60. Read pages 239-246 of [CLO].

61. Do Exercise 4 on page 247 of [CLO], which is.... Let $V = \mathcal{V}(y - x^n, z - x^m)$, where $m, n \in \mathbb{N}$. Show that V is isomorphic as a variety to $\Bbbk$ by constructing explicit inverse regular functions $\phi : \Bbbk \to V$ and $\chi : V \to \Bbbk$.

62. (a) Consider $\langle x \rangle \subset \mathbb{C}[x]$ and write $V = \mathcal{V}(x) \subset \mathbb{C}$.
    (i) Find $|V|$.
    (ii) Find $\dim_{\mathbb{C}}(\mathbb{C}[V])$. (Hint: recall $\mathbb{C}[V] = \{f|_V : f \in \mathbb{C}[x]\}$.)
    (iii) Find $\dim_{\mathbb{C}}\left(\dfrac{\mathbb{C}[x]}{\langle x \rangle}\right)$ directly, then compare with (ii) (for the comparison, recall Theorem 7 on pg 224 of [CLO]).
    (b) Consider $\langle x^2 \rangle \subset \mathbb{C}[x]$ and write $V = \mathcal{V}(x^2) \subset \mathbb{C}$.
    (i) Find $|V|$.
    (ii) Find $\dim_{\mathbb{C}}(\mathbb{C}[V])$. (Hint: recall $\mathbb{C}[V] = \{f|_V : f \in \mathbb{C}[x]\}$.)
    (iii) Find $\dim_{\mathbb{C}}\left(\dfrac{\mathbb{C}[x]}{I(V)}\right)$ directly, then compare with (ii) (for the comparison, recall Theorem 7 on pg 224 of [CLO]).
    (iv) Find $\dim_{\mathbb{C}}\left(\dfrac{\mathbb{C}[x]}{\langle x^2 \rangle}\right)$ directly.

63. This question is a variant of Exercise 9 on pg 257 of [CLO]. Let V denote an irreducible variety and let $f \in \Bbbk(V)$. If we write $f = g/h$, where $g, h \in \Bbbk[V]$, then we know that f is defined on $V \setminus \mathcal{V}_V(h)$. However, as discussed in class, f might make sense on a larger set. We will illustrate this phenomenon in this question by considering $V = \mathcal{V}(xz - yw) \subset \mathbb{C}^4$.
    (a) You may assume that $xz - yw$ is irreducible in $\mathbb{C}[x, y, z, w]$ and that $\langle xz - yw \rangle$ is a prime ideal in $\mathbb{C}[x, y, z, w]$. Show that it follows that V is irreducible and that $I(V) = \langle xz - yw \rangle$.
    (b) Let $f = x/y \in \mathbb{C}(V)$, so that f is defined on $V \setminus \mathcal{V}_V(y)$. Show that $\mathcal{V}_V(y)$ is the union of planes $\mathcal{V}(x, y) \cup \mathcal{V}(y, z)$.
    (c) Show that $f = w/z$ in $\mathbb{C}(V)$, and conclude that f is defined on $V \setminus \mathcal{V}(y, z)$, so that f has domain larger than originally thought in (b). (Note that what makes this possible is that we have two fundamentaly different ways of representing the rational function f. This is one reason why rational functions are subtle to deal with.)

64. Read pages 357-364 of [CLO].

65. Let $\Bbbk = \mathbb{R}$ and consider the hyperbola $\mathcal{V}(xy - 1) \subset \mathbb{R}^2$.
    (a) Homogenize the defining polynomial to get a new variety $\mathcal{V}(f) \subset \mathbb{P}^2$.
    (b) Recognise the original hyperbola as a subset V of $\mathcal{V}(f)$.
    (c) Find $\mathcal{V}(f) \setminus V$, and describe it.
    (d) Explain what $\mathcal{V}(f) \setminus V$ is describing intuitively in the $\mathbb{R}^2$ picture.

66. Let $\Bbbk = \mathbb{R}$ and consider the vertical line $\mathcal{V}(x - c) \subset \mathbb{R}^2$, where $c \in \mathbb{R}^\times$.
    (a) Homogenize the defining polynomial to get a new variety $\mathcal{V}(f) \subset \mathbb{P}^2$.
    (b) Recognise the original vertical line as a subset V of $\mathcal{V}(f)$.
    (c) Find $\mathcal{V}(f) \setminus V$, and describe it.
    (d) Explain what $\mathcal{V}(f) \setminus V$ is describing intuitively in the $\mathbb{R}^2$ picture.

67. Let $\Bbbk$ be an algebraically closed field. Show that every homogeneous polynomial in two variables with coefficients in $\Bbbk$ can be completely factored into a product of linear homogeneous polynomials.

68. Do Exercise 5 on page 366 of [CLO].

69. Do Exercise 11(a)(b) on page 367 of [CLO].

70. Read pages 368-375 of [CLO].

71. Do Exercise 11 on page 376 of [CLO].

72. Do Exercise 12 on page 376 of [CLO].

73. Read pages 379-385 of [CLO].

74. A homogeneous ideal is said to be *prime* if it is prime as an ideal in $k[x_0, \ldots, x_n]$.
    (a) Prove that a homogeneous ideal $J \subset k[x_0, \ldots, x_n]$ is prime if and only if whenever the product of two *homogeneous* polynomials f, g satisfies $fg \in J$, then $f \in J$ or $g \in J$.
    (b) Let J denote a homogeneous ideal. Show that the projective variety $\mathcal{V}(J)$ is irreducible if and only if $I(\mathcal{V}(J))$ is prime.

75. Read page 387 of [CLO].

76. Prove that if an affine variety V is irreducible then its projectivization $\mathbb{P}(V)$ is irreducible. (Hint: use $\mathbb{P}(V)$ is the smallest projective variety containing V.)

77. Prove that $I(\mathbb{P}(V)) = {}^h\!I(V)$ for any affine variety $V \subseteq \mathbb{A}^n$. (Hint: adapt the proof of (b) in Proposition on $\mathbb{P}(V)$ (or use another method).)

78. Do Exercise 9 on page 392 of [CLO], which is.... Prove that an affine variety V is irreducible if and only if its projectivization $\mathbb{P}(V)$ is irreducible. (Hint: use [CLO, Page 392, #7] without proof.) (Remark: this question extends Question 76.)

79. Do Exercise 14(a)(c) on page 393 of [CLO], which is.... In this exercise, we will see that $\mathbb{P}^n \times \mathbb{P}^m$ can be identified with a certain projective variety in $\mathbb{P}^{n+m+mn}$ known as a *Segre variety*. The idea is as follows. Let $p = (x_0, \ldots, x_n)$ be homogeneous coordinates of $p \in \mathbb{P}^n$ and let $q = (y_0, \ldots, y_m)$ be homogeneous coordinates of $q \in \mathbb{P}^m$. The Segre mapping $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{n+m+mn}$ is defined by taking the pair $(p, q) \in \mathbb{P}^n \times \mathbb{P}^m$ to the point in $\mathbb{P}^{n+m+mn}$ with homogeneous coordinates

$$(x_0y_0, x_0y_1, \ldots, x_0y_m, x_1y_0, \ldots, x_1y_m, \ldots, x_ny_m),$$

the coordinates being all the possible products x_iy_j where $0 \le i \le n$ and $0 \le j \le m$. The image of σ is a projective variety called a *Segre variety*.

(a) Show that σ is well defined. (Hint: show that we obtain the same point in $\mathbb{P}^{n+m+mn}$ no matter what homogeneous coordinates we use for p and q.)

(b) Taking $n = m = 1$, write out $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ explicitly and find homogeneous equations for the image. (Hint: you should show that a single quadratic equation suffices – this Segre variety is a *quadric surface* in $\mathbb{P}^3$.)

80. Assume that $\Bbbk = \mathbb{C}$. Find the singular points and the tangent lines at the singular points for the following affine varieties in $\mathbb{A}^2 = \mathbb{C}^2$:
(a) $\mathcal{V}(y^2 - x^3 + x)$ (b) $\mathcal{V}((x^2+y^2)^2 + 3x^2y - y^3)$ (c) $\mathcal{V}((x^2+y^2)^3 - 4x^2y^2)$.

81. Assume that $\Bbbk = \mathbb{C}$. Given an affine variety V, the tangent space to V at the origin $p \in V$ is defined to be
$$T_p(V) = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{A}^n : \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(p)\alpha_i = 0 \text{ for all } g \in I(V)\}.$$
If $0 \neq f \in \mathbb{C}[x_1, \ldots, x_n]$ is irreducible and if $\mathcal{V}(f)$ contains the origin p, prove that
$$T_p(\mathcal{V}(f)) = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{A}^n : \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)\alpha_i = 0\}.$$

82. Let $f = y - x^3$, $g = y \in \mathbb{C}[x, y]$. Homogenize f and g using a new variable, z, and order the variables $x,\ y,\ z$. Use U_x for U_0, U_y for U_1, and U_z for U_2. You should find that $\mathcal{V}({}^hf) \cap \mathcal{V}({}^hg) = \{(0,0,1)\} \subset \mathbb{P}^2$. Hence, $U_x \cap \mathcal{V}({}^hf) \cap \mathcal{V}({}^hg) = \emptyset$, so $i(\mathcal{V}({}^hf) \cap \mathcal{V}({}^hg),\ p) = 0$ for all $p \in U_x$. Verify this is true by applying the definition of $i(\mathcal{V}({}^hf) \cap \mathcal{V}({}^hg),\ p)$ using $U_j = U_x$.

83. Let $f = x^2 - y^2 - 1$, $g = y - x \in \mathbb{C}[x, y]$. Show that $i(\mathcal{V}({}^hf) \cap \mathcal{V}({}^hg),\ p) = 2$ for all $p \in \mathcal{V}({}^hf) \cap \mathcal{V}({}^hg)$.

84. Find $X = \mathrm{Spec}R$, where R is:
(a) $\mathbb{Z}$ (b) $\mathbb{Z}/3\mathbb{Z}$ (c) $\mathbb{Z}/6\mathbb{Z}$ (d) $\mathbb{C}[x]$ (e) $\mathbb{C}[x]/\langle x^2 \rangle$ (f) $\mathbb{C}[x]/\langle x^3 - x^2 \rangle$.

85. Which points of X are closed and which are open, where X is:
(a) $\mathrm{Spec}\mathbb{C}[x]$ (b) $\mathrm{Spec}\mathbb{C}[x]/\langle x^2 \rangle$ (c) $\mathrm{Spec}\mathbb{C}[x]/\langle x^3 - x^2 \rangle$?

86. Let $R = \dfrac{\mathbb{C}[x,\ y]}{\langle x^2 y \rangle}$. Find $X = \mathrm{Spec}R$, its closed sets and its open sets; find $R[x^{-1}]$ and $R[y^{-1}]$.

87. Let $S = \dfrac{\mathbb{C}[x,\ y]}{\langle x^2 y \rangle}$. Find $X = \mathrm{Proj}S$, its closed sets and its open sets.

88. Complete the student feedback survey online. (It needs to be completed by 11 pm Wed Dec 4.) Thank you!

END HOMEWORK