In the following, unless otherwise specified, $\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$, and V denotes any vector space over $\mathbb{F}$. You should attempt all the questions, justifying your answers, but $*$ denotes the questions that are the minimum to be attempted. Page numbers refer to Axler's book, 2nd Ed. All questions are potential questions for the tests, unless otherwise stated.

1. $*$ Read Chapter 1.
2. Do Exercise 1 on page 19.
3. Do Exercise 2 on page 19.
4. $*$ From page 9, if $v \in V$, what is the definition of $-v$?
5. $*$ Do Exercise 3 on page 19.
6. $*$ Do Exercise 5 on page 19.
7. $*$ If U_1 and U_2 are subspaces of V, prove that $U_1 \cap U_2$ is also a subspace of V.
8. Do Exercise 10 on page 19.
9. $*$ Do Exercise 14 on page 20.
10. Read Chapter 2.
11. Prove that the set of $n \times n$ matrices, $M_n(\mathbb{F})$, as a vector space over $\mathbb{F}$, has dimension n^2. What is the dimension of the set of $m \times n$ matrices as a vector space over $\mathbb{F}$?
12. $*$ Prove that if $\{v_1, v_2\} \subseteq V$ is linearly dependent, then $v_1 \in \mathbb{F}v_2$ or $v_2 \in \mathbb{F}v_1$.
13. Do Exercise 2 on page 35.
14. $*$ Do Exercise 4 on page 35.
15. $*$ Do Exercise 8 on page 35.
16. Do Exercise 9 on page 35.
17. $*$ Do Exercise 10 on page 35.
18. $*$ Do Exercise 11 on page 36.
19. $*$ Do Exercise 14 on page 36.
20. Do Exercise 17 on page 36.
21. Read Chapter 3.

22. Let $\{v_1, v_2, \ldots, v_n\}$ be a subset of a vector space V. Prove that the map $T : \mathbb{F}^n \to V$, given by $T((\alpha_1, \ldots, \alpha_n)) = \alpha_1 v_1 + \cdots + \alpha_n v_n$, is a linear map.

23. $*$ Do Exercise 1 on page 59.

24. $*$ Do Exercise 3 on page 59 (hint: 0 is a nice function!).

25. $*$ Do Exercise 4 on page 59.

26. $*$ Do Exercise 5 on page 59.

27. $*$ Do Exercise 9 on page 59.

28. $*$ Do Exercise 10 on page 60.

29. Do Exercise 11 on page 60. (Why is Theorem 3.4 not applicable to this question?)

30. Do Exercise 12 on page 60.

31. Do Exercise 16 on page 60.

32. Find all linear maps $T : \mathbb{R}^2 \to \mathbb{R}^2$ which map the line $y = x$ to the line $y = 5x$.

33. Find a linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(\mathbb{R}^2) = \mathbb{R}e_2 \subset \mathbb{R}^3$ with the property that $\mathrm{Ker}(T) = \mathbb{R}(2, 3)$.

34. $*$ Do Exercise 21 on page 61 (hint: consider $[\,T(e_1) \;\; T(e_2) \;\; \ldots \;\; T(e_n)\,]$).

35. Do Exercise 22 on page 61.

36. Do Exercise 23 on page 61.

37. $*$ Do Exercise 26 on page 62.

38. $*$ Read Chapter 4 – at least the chit-chat & statements of results, if not the proofs. (We will not cover this chapter in class time, but we will use it in later chapters.)

39. $*$ Read the handout on writing proofs.

40. $*$ Read Chapter 5.

41. $*$ Do Exercise 1 on page 94.

42. $*$ Do Exercise 4 on page 94.

43. $*$ Do Exercise 5 on page 94.

44. $*$ Do Exercise 6 on page 94.

45. $*$ Do Exercise 7 on page 94.

46. $*$ Do Exercise 8 on page 94.

47. Do Exercise 18 on page 95.

48. Do Exercise 19 on page 95.

49. Do Exercise 23 on page 96.

50. Do Exercise 9 on page 94.

51. $*$ Do Exercise 10 on page 94.

52. Do Exercise 12 on page 95.

53. $*$ Do Exercise 14 on page 95.

54. (This question is optional and is for those who like a challenge!) Do Exercise 15 on page 95.

55. Let $T \in \mathcal{L}(\mathbb{R}^2)$ be defined by $T(\alpha, \beta) = (-\beta, \alpha)$. The eigenvalues of T are given on page 78. Find a polynomial $p \in \mathcal{P}(\mathbb{R})$ such that $p(T)$ has an eigenvalue $\mu \in \mathbb{R}$, yet $\mu \neq p(\lambda)$ for any $\lambda \in \mathbb{R}$. Compare your example with the statement of Exercise 15 on page 95.

56. $*$ Do Exercise 20 on page 95.

57. $*$ Do Exercise 21 on page 95.

58. $*$ Read Chapter 6.

59. Do Exercise 4 on page 122.

60. $*$ Do Exercise 6 on page 122.

61. Do Exercise 9 on page 123.

62. $*$ Do Exercise 10 on page 123.

63. $*$ Do Exercise 14 on page 124.

64. $*$ Do Exercise 15 on page 124.

65. $*$ Do Exercise 16 on page 124.

66. $*$ Do Exercise 21 on page 124.

67. Do Exercise 22 on page 124.

68. $*$ Do Exercise 26 on page 125.

69. $*$ Do Exercise 27 on page 125.

70. $*$ Do Exercise 28 on page 125.

71. $*$ Do Exercise 29 on page 125.

72. $*$ Do Exercise 30 on page 125.

73. $*$ Do Exercise 31 on page 125.

74. Do Exercise 32 on page 125.

75. $*$ Suppose that V and W are inner-product spaces. Recall that if $T \in \mathcal{L}(V,W)$, then its adjoint is $T^* \in \mathcal{L}(W,V)$. Define $\bar{T} : W^* \to V^*$ by $\bar{T}(f) = f \circ T$, where $f \in W^*$. If $f = \langle __, w \rangle_W$ with $w \in W$, prove that $\bar{T}(f) = \langle __, T^*(w) \rangle_V$.

76. $*$ $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$. Find the dual basis for $(\mathbb{R}^3)^*$ relative to $\mathcal{B}$.

77. Let $W \subseteq V$ be a subspace of V and let $W^\perp \subseteq V^*$ denote

$$W^\perp = \{f \in V^* : f(w) = 0 \text{ for all } w \in W\}.$$

(Note this $W^\perp$ is not the $W^\perp$ defined in class!) Prove that $W^\perp$ is a subspace of V^*. Assuming $\dim(V) < \infty$, prove $\dim(W) + \dim(W^\perp) = \dim(V)$. (Hint: start with a basis for W, & extend it to a basis for V, then use a dual basis for V^*.)

78. Let $v \in V$ and let $F_v : V^* \to \mathbb{F}$ be defined by $F_v(f) = f(v)$ for all $f \in V^*$.

(a) Prove that F_v is a function and $F_v \in (V^*)^*$.

(b) Prove that the map $\phi : v \mapsto F_v$ is well defined and is a linear map from V to $(V^*)^*$.

(c) Assume that V is finite-dimensional. Prove that the map ϕ in (b) is an isomorphism.

79. $*$ For each of the following quadratic forms Q, find a symmetric matrix A such that $Q(v) = v^T A v$ for all $v \in V$.

(a) $Q(x_1, x_2) = 6x_1^2 - 7x_1x_2 + 8x_2^2$.

(b) $Q(x_1, x_2) = x_1x_2$.

(c) $Q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 6x_1x_3 + 7x_2x_3$.

80. $*$ For each of the following symmetric matrices, give the corresponding quadratic form.

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$.

81. $*$ Read Chapter 7.

82. $*$ Do Exercise 1 on page 158. (Hint: consider $T(1)$ & $T(x)$.)

83. Do Exercise 6 on page 158.

84. $*$ Do Exercise 8 on page 159.

85. $*$ Do Exercise 9 on page 159.

86. $*$ Do Exercise 10 on page 159.

87. $*$ Do Exercise 11 on page 159.

88. Do Exercise 13 on page 159.

89. Do Exercise 15 on page 159.

90. $*$ Read the chit-chat & results in Chapter 8.

91. Read pages 177-8.

92. $*$ Do Exercise 1 on page 188.

93. $*$ Do Exercise 2 on page 188.

94. $*$ Do Exercise 3 on page 188.

95. Do Exercise 5 on page 188.

96. $*$ Do Exercise 6 on page 188.

97. Do Exercise 7 on page 188.

98. Do Exercise 10 on page 189.

99. $*$ Do Exercise 14 on page 189.

100. $*$ Do Exercise 15 on page 189.

101. $*$ Find the minimal polynomial of each of the following matrices:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Factor the minimal polynomials you have found, and, in each case, describe the minimal polynomial in terms of the appearance of the matrix. What is the pattern?

102. $*$ Repeat Question 101 for the following matrices:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

103. $*$ Repeat Question 101 for the following matrices: $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$

104. Do Exercise 17 on page 189.

105. Do Exercise 20 on page 190.

106. $*$ Do Exercise 23 on page 190.

107. $*$ Assume $\mathbb{F} = \mathbb{C}$. Let $\dim(V) < \infty$ and let $T \in \mathcal{L}(V)$ be diagonalizable. Prove that if $U \subseteq V$ is an invariant subspace, then $T|_U$ is diagonalizable. (Hint: use the previous question.)

108. $*$ Assume $\mathbb{F} = \mathbb{C}$. Let $\dim(V) < \infty$ and let $T,\ S \in \mathcal{L}(V)$ where T and S are diagonalizable (possibly with respect to different bases). Prove that there exists a basis for V with respect to which both T and S have diagonal matrices if and only if T and S commute on V. (Hint: the previous question should be useful.) In this case, T and S are said to be *simultaneously diagonalizable*.

109. $*$ Assume $\mathbb{F} = \mathbb{C}$. Find the Jordan normal (canonical) form of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

and find an invertible matrix P such that $P^{-1}AP$ is the Jordan normal form of A.

110. $*$ Assume $\mathbb{F} = \mathbb{C}$. Find the Jordan normal (canonical) form of the matrix

$$A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$$

and find an invertible matrix P such that $P^{-1}AP$ is the Jordan normal form of A. (Hint: refer to the model example on the class handout.)

111. $*$ Assume $\mathbb{F} = \mathbb{C}$. Find the Jordan normal (canonical) form of the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and find an invertible matrix P such that $P^{-1}AP$ is the Jordan normal form of A. (Hint: refer to the model example on the class handout.)

112. $*$ Matrices A and B are said to be *similar* if there exists an invertible matrix Q such that $A = Q^{-1}BQ$. Assume $\mathbb{F} = \mathbb{C}$. Find the Jordan normal (canonical) form of the following pairs of matrices and state which are similar.

(a)
$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(b) (Here $i^2 = -1$.)
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

(c)
$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

113. $*$ Let $T \in \mathcal{L}(V)$ and let $\mathcal{B}_1$ and $\mathcal{B}_2$ denote two bases on V. Let $A_1 = M(T,\ \mathcal{B}_1,\ \mathcal{B}_1)$ and let $A_2 = M(T,\ \mathcal{B}_1,\ \mathcal{B}_2)$.

(a) Find an example with $V = \mathbb{F}^2$ where A_1 and A_2 do <u>not</u> have the same eigenvalues.

(b) Why does (a) not contradict the fact that similar matrices have the same eigenvalues?

114. $*$ Assume $\mathbb{F} = \mathbb{C}$. Find the Jordan normal (canonical) form of the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ -3 & 3 & 2 & 8 & -2 & 0 & -1 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 \end{bmatrix},$$

and find an invertible matrix P such that $P^{-1}AP$ is the Jordan normal (canonical) form of A. (Hint: refer to the model example on the class handout.)

115. $*$ Assume $\mathbb{F} = \mathbb{R}$. Find the Jordan normal (canonical) form of the matrix

$$A = \begin{bmatrix} 37 & 10 & -5 & 5 \\ 10 & 22 & 10 & -10 \\ -5 & 10 & 37 & 5 \\ 5 & -10 & 5 & 37 \end{bmatrix},$$

and find an orthogonal matrix P such that P^TAP is the Jordan normal (canonical) form of A. (Hint: P is orthogonal if P is invertible and $P^{-1} = P^T$; notice that A is symmetric (self-adjoint), and apply the Gram-Schmidt orthogonalization procedure to each eigenspace of A in turn.)

116. Read pages 225-236. Learn definition 10.25 on page 229.

117. Complete online the student feedback survey for this class. Check your mymav email for the link. Thanks!!

END HOMEWORK