GRADED SKEW CLIFFORD ALGEBRAS THAT ARE TWISTS OF GRADED CLIFFORD ALGEBRAS

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Abstract. In 2010, a quantized analog of a graded Clifford algebra (GCA), called a graded skew Clifford algebra (GSCA), was proposed by Cassidy and Vancliff, and many properties of GCAs were found to have counterparts for GSCAs. In particular, a GCA is a finite module over a certain commutative subalgebra C, while a GSCA is a finite module over a (typically non-commutative) analogous subalgebra R. We consider the case that a regular GSCA is a twist of a GCA by an automorphism, and we prove, in this case, R is a skew polynomial ring and a twist of C by an automorphism.

Introduction

By 2011, in [5, 10], it had been proved that almost all quadratic regular algebras of global dimension two or three may be classified using certain non-commutative algebras called graded skew Clifford algebras (GSCAs). The latter algebras were first defined by Cassidy and Vancliff in [5], and may be viewed as a quantized analog of a graded Clifford algebra (GCA). Many properties of GCAs were found in [5] to have counterparts for GSCAs; in particular, a GCA is quadratic and regular if and only if its associated quadric system has no base points, whereas a GSCA is quadratic and regular if and only if its associated (non-commutative) quadric system is normalizing

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and has no base points (see Theorem 1.3). Moreover, a GCA is a finite module over a certain commutative subalgebra C, while a GSCA is a finite module over a (typically non-commutative) analogous subalgebra R. In this article, we consider the case that a regular GSCA A is a twist by an automorphism of a GCA B. In this setting, we prove, in Theorem 2.4, that R is a skew polynomial ring and is a twist of C by an automorphism. We also demonstrate in Example 2.1 that this can fail if A is not a twist of a GCA.

This article consists of two sections: in Section 1, notation and terminology are defined, while Section 2 is devoted to proving our main result, which is given in Theorem 2.4.

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1. Definitions

In this section, we introduce the algebras to be discussed in the paper, and some known results concerning them, including the connection between their homological properties and certain associated geometric data.

Throughout the article, $\Bbbk$ denotes an algebraically closed field such that $\mathrm{char}(\Bbbk) \neq 2$, and $M(n, \Bbbk)$ denotes the vector space of $n \times n$ matrices with entries in $\Bbbk$. If M is a matrix, then M_{ij} will denote the ij'th entry of M. For a graded $\Bbbk$-algebra E, the span of the homogeneous elements in E of degree i will be denoted E_i, and if F is any ring or vector space, then $F^\times$ will denote the nonzero elements in F.

Let $N_1, \ldots, N_n \in M(n, \Bbbk)$ denote symmetric matrices. By definition (c.f., [7]), a graded Clifford algebra (GCA) is the $\Bbbk$-algebra B on degree-one generators $X_1, \ldots, X_n$ and on degree-two generators $Y_1, \ldots, Y_n$ with defining relations given by

(a) $X_iX_j + X_jX_i = \sum_{k=1}^{n} (N_k)_{ij}\, Y_k$ for all $i, j = 1, \ldots, n$, and

(b) Y_k central for all $k = 1, \ldots, n$.

We write C for the subalgebra of B generated by $Y_1, \ldots, Y_n$, and note that C is a polynomial ring and that B is a finite module over C. Results on GCAs can be found in [11, 12].

The notion of graded skew Clifford algebra is similarly defined, but uses a generalization of the notion of symmetric matrix as follows. Let $\mu \in M(n, \Bbbk)$, where $\mu_{ij}\mu_{ji} = 1$ for all $i,\ j$ with $i \neq j$. In [5], a matrix $M \in M(n, \Bbbk)$ is defined to be μ-symmetric if $M_{ij} = \mu_{ij}M_{ji}$ for all $i,\ j$.

Definition 1.1. [5] Suppose, additionally, that $\mu_{ii} = 1$ for all i and let $M_1, \ldots, M_n \in M(n, \Bbbk)$ be μ-symmetric matrices. A graded skew Clifford algebra (GSCA) is the $\Bbbk$-algebra A on degree-one generators $x_1, \ldots, x_n$ and on degree-two generators $y_1, \ldots, y_n$ with defining relations given by

(a) degree-two relations: $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^{n} (M_k)_{ij}\, y_k$ for all $i, j = 1, \ldots, n$, and

(b) degree-three and degree-four relations that guarantee the existence of a normalizing sequence $\{y_1', \ldots, y_n'\}$ that spans $\sum_{k=1}^{n} \Bbbk y_k$.

We refer to the subalgebra of A generated by $y_1, \ldots, y_n$ as R, and note that A is a finite module over R.

Clearly, a GCA is a special case of a GSCA. It is proved in [5] that it is possible for all the y_k to belong to $(A_1)^2$, and this happens if and only if $M_1, \ldots, M_n$ are linearly independent. However, even in this case, A need not be a quadratic algebra.

One may associate to a symmetric matrix a quadratic form and hence a quadric in $\mathbb{P}^{n-1}$. Similarly, as was shown in [5], one may associate a (non-commutative) quadratic form and a (non-commutative) quadric to a μ-symmetric matrix as follows. Let S denote the $\Bbbk$-algebra on generators $z_1, \ldots, z_n$ with defining relations $z_j z_i = \mu_{ij} z_i z_j$ for all $i,\ j$, where $\mu_{ii} = 1$ for all i. If $z = (z_1, \ldots, z_n)^T$ and if $M \in M(n, \Bbbk)$ is a μ-symmetric matrix, then the map $M \to z^T M z$ from the vector space of μ-symmetric $n \times n$ matrices to S_2 is an isomorphism of vector spaces ([5]). As in [5], we call the elements of S_2 quadratic forms. If $q \in S_2$, then the intersection in $\mathbb{P}(S_1^*) \times \mathbb{P}(S_1^*)$ of the zero locus of the defining relations of S with the zero locus of q is called the quadric associated to q. If $q \in S_2$ is normal in S, then its quadric parametrizes those point modules over S that are annihilated by q; thus, this notion of quadric generalizes the commutative definition.

Definition 1.2.

(a) [5] The span of quadratic forms $q_1, \ldots, q_m \in S_2$ will be called the *quadric system* associated to $q_1, \ldots, q_m$. If a quadric system is given by a normalizing sequence in S, then it is called a *normalizing quadric system*.

(b) [6] We define a *left base point* of a quadric system $\{\mathfrak{Q}\} \subset S_2$ to be any left base-point module over $S/\langle \mathfrak{Q} \rangle$; that is, to be any 1-critical graded left module over $S/\langle \mathfrak{Q} \rangle$ that is generated by its homogeneous degree-zero elements and which has Hilbert series $H(t) = c/(1-t)$, for some $c \in \mathbb{N}$. We say a quadric system is *left base-point free* if it has no left base points. Similarly, for right base point, etc.

If S is commutative, then the notions of "base point" and "base-point free" agree with their commutative counterparts, since the only base-point modules in this case are point modules.

By [6, Corollary 11], a normalizing quadric system $\mathfrak{Q}$ is left base-point free if and only if $\dim_{\Bbbk}(S/\langle \mathfrak{Q} \rangle) < \infty$. Hence, such a quadric system is left base-point free if and only if it is right base-point free. In particular, the adjectives "right" and "left" may be dropped when referring to a *normalizing* quadric system being base-point free.

This geometric data associated to a GCA or GSCA has fundamental influence on homological data of the algebra as demonstrated in the next result; the reader is referred to [1, 8] for definitions of the terms.

Theorem 1.3.

(a) [4, 7] *The GCA B is quadratic, Auslander-regular of global dimension n and satisfies the Cohen-Macaulay property with Hilbert series $1/(1-t)^n$ if and only if its associated (commutative) quadric system is base-point free; in this case, B is AS-regular and is a noetherian domain.*

(b) [5, 6] *The GSCA A is quadratic, Auslander-regular of global dimension n and satisfies the Cohen-Macaulay property with Hilbert series $1/(1-t)^n$ if and only if its associated quadric system is normalizing and base-point free; in this case, A is AS-regular, is a noetherian domain and is unique up to isomorphism.*

In the next section, we will consider a GSCA that is a twist (in the sense of Definition 1.4) of a GCA by an automorphism.

Definition 1.4. [3, §8] Let $D = \bigoplus_{k\geq 0} D_k$ be a graded $\Bbbk$-algebra and let ϕ be a graded degree-zero automorphism of D. The twist D' of D by ϕ is a graded $\Bbbk$-algebra that is the vector space $\bigoplus_{k\geq 0} D_k$ with a new multiplication $*$ defined as follows: if $a' \in D'_i = D_i,\ b' \in D'_j = D_j$, then $a' * b' = (a\phi^i(b))'$, where the right-hand side is computed using the original multiplication in D and a is the image of a' in D, etc. The twist of a quadratic algebra is again a quadratic algebra.

For ϕ and a as in Definition 1.4, we will write a^ϕ for $\phi(a)$.

We close this section with a simple lemma concerning GCAs that will be useful in the next section.

Lemma 1.5. *Let B be a GCA as above. If $a,\ b \in B_1$, then $ab + ba$ is central in B.*

Proof. The result is a consequence of $X_iX_j + X_jX_i$ being central in B for all $i,\ j$. ■

2. The Main Theorem

In this section, we compare the subalgebras R and C that are defined in Section 1. We prove in Theorem 2.4 that if the GSCA A is a twist by an automorphism of a regular GCA B, then R is a twist of C by an automorphism and is a skew polynomial ring (that is, R is a domain on n generators with $n(n-1)/2$ defining relations that guarantee that each generator is normal in R).

Not surprisingly, the algebra R is not, in general, a skew polynomial ring nor a twist of a polynomial ring, and we first demonstrate this via a simple example.

Example 2.1. [9, §3.2] Let $n = 3$, $\mu \in M(3, \Bbbk)$ be as above and let

$$M_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

where $\mu_{13} = \mu_{23} = 1$ and $\mu_{12} = 2$. By Theorem 1.3(b), the GSCA A associated to this data is quadratic and regular, and so is the $\Bbbk$-algebra on x_1, x_2, x_3 with defining relations

$$x_1x_2 + 2x_2x_1 = x_3^2, \qquad x_1x_3 + x_3x_1 = 0, \qquad x_2x_3 + x_3x_2 = 0,$$

where $y_i = x_i^2$ for $i = 1,\ 2,\ 3$. By Lemma 2.2 (below), since $\mu_{13} \neq \mu_{12}\mu_{23}$, the associated algebra S is not a twist of a polynomial ring and so, by [5, Proposition 4.5], A is not a twist of a GCA by an automorphism. Moreover, y_3 is central in R, but no other element in $\sum_{k=1}^{3} y_k$ is normal in R (this can be seen by using a computer-algebra program such as W. Schelter's Affine program and noting that any normal element in R would be normal in $R/\langle y_3 \rangle$ in order to simplify the computations involved). It follows that R is not a skew polynomial ring. Moreover, there is an insufficient number of relations in low degree amongst the y_k for R to be a twist of a polynomial ring.

Lemma 2.2. *Let $\mu \in M(n, \Bbbk)$ and S be as in Section 1. The algebra S is a twist of the polynomial ring $K = \Bbbk[Z_1, \ldots, Z_n]$ by a graded automorphism $\sigma \in Aut(K)$ of degree zero if and only if $\mu_{ik} = \mu_{ij}\mu_{jk}$ for all $i,\ j,\ k$; in this case, $\sigma|_{K_1}$ is semisimple, and, for all $i,\ j$, we have $\mu_{ij} = \rho_i/\rho_j$, where $\rho_i \in \Bbbk^\times$ and $\sigma(Z_i) = \rho_i Z_i$ for all i.*

Proof. The first part of the result follows from [2], since $\mu_{ik} = \mu_{ij}\mu_{jk}$ for all $i,\ j,\ k$ if and only if the point scheme of S (or the zero locus of the defining relations of S) is isomorphic to $\mathbb{P}^{n-1}$, and the latter holds if and only if S is a twist of the polynomial ring on n variables by an automorphism.

For the second part of the result, suppose S is a twist of the polynomial ring $K = \Bbbk[Z_1, \ldots, Z_n]$ by a graded automorphism $\sigma \in \mathrm{Aut}(K)$ of degree zero, where we identify z_i and Z_i for all i. From the relations in S, we have

$$Z_j Z_i^\sigma = \mu_{ij} Z_i Z_j^\sigma \quad \text{in } K \tag{$*$}$$

for all $i,\ j$. However, K is a commutative unique factorization domain and $\deg(Z_i) = 1$ for all i, so Z_i is irreducible in K. Moreover, if $i \neq j$, then $Z_i \nmid Z_j$, so, by $(*)$, $Z_i | Z_i^\sigma$ for all i. Since Z_i^σ has degree one, $Z_i^\sigma \in \Bbbk^\times Z_i$ for all i. Hence, $\sigma|_{K_1}$ is semisimple. Writing $Z_i^\sigma = \rho_i Z_i$, where $\rho_i \in \Bbbk^\times$ for all i, and substituting into $(*)$ completes the proof. ∎

Remark 2.3. Suppose that B is a regular GCA (in the sense of Theorem 1.3) and that A is a GSCA that is a twist of B by a graded automorphism $\tau \in \mathrm{Aut}(B)$ of degree zero. As was shown in Section 1, there is a skew polynomial ring S associated to A. By [5, Proposition 4.5], since A is a twist of B by τ, there exists a choice for S so that S is a twist of the polynomial ring

$K = \Bbbk[Z_1, \ldots, Z_n]$ by τ^{-1} and conversely. By Lemma 2.2, $\tau|_{K_1}$ is semisimple and, for each i, we have $\tau(Z_i) = \lambda_i Z_i$ for some $\lambda_i \in \Bbbk^\times$ and $\mu_{ij} = \lambda_j/\lambda_i$ for all $i,\ j$. (In the notation of Lemma 2.2, $\lambda_i = \rho_i^{-1}$ for all i, since $\tau = \sigma^{-1}$.)

Theorem 2.4. *Suppose that A is a regular GSCA in the sense of Theorem 1.3(b) and that R is the subalgebra of A generated by the y_k as in Definition 1.1. If A is a twist of a GCA B by a graded automorphism $\tau \in Aut(B)$ of degree zero, then R is a twist of the analogous subalgebra C of B generated by the Y_k and is a skew polynomial ring (that is, R is a domain that has exactly $n(n-1)/2$ defining relations that guarantee that each y_k is normal in R).*

Proof. By Remark 2.3, we may assume that S is a twist of the polynomial ring $K = \Bbbk[Z_1, \ldots, Z_n]$ by τ^{-1}, and that $\tau(Z_i) = \lambda_i Z_i$ for some $\lambda_i \in \Bbbk^\times$ and that $\mu_{ij} = \lambda_j/\lambda_i$ for all $i,\ j$. If necessary, by Lemma 1.5, we may also re-choose the $X_k \in B_1$ so that $\{X_1, \ldots, X_n\}$ is dual to the basis $\{Z_1, \ldots, Z_n\}$ for K_1 and so that the degree-two relations of B still have the form given in Section 1 (although the symmetric matrices $N_1, \ldots, N_n$ might change). With this choice of bases, it follows that $X_i^\tau = \lambda_i X_i$ for all i, and that the twist of X_i is x_i. Hence,

$$x_i x_j + \mu_{ij} x_j x_i = x_i x_j + (\lambda_j/\lambda_i) x_j x_i \in \Bbbk^\times (x_i^\tau x_j + x_j^\tau x_i) \qquad (**)$$

for all $i,\ j$. We will prove that each $x_i^\tau x_j + x_j^\tau x_i = r_{ij}$ is a normal element of A. By Definition 1.1 and $(**)$, $r_{ij} \in R$ for all $i,\ j$, so the subalgebra of A generated by the r_{ij} is contained in R. Since A is quadratic, each y_k is a function of the r_{ij} and so R is the subalgebra of A generated by the r_{ij}. Moreover, for all $i,\ j,\ k$, we have

$$\begin{aligned} x_k r_{ij} &= x_k(x_i^\tau x_j + x_j^\tau x_i) \\ &= X_k(X_i^{\tau^2} X_j^{\tau^2} + X_j^{\tau^2} X_i^{\tau^2}) \\ &= \lambda_i^2 \lambda_j^2 X_k (X_i X_j + X_j X_i) \\ &= \lambda_i^2 \lambda_j^2 (X_i X_j + X_j X_i) X_k \\ &= \lambda_k^{-2} \lambda_i^2 \lambda_j^2 (x_i^\tau x_j + x_j^\tau x_i) x_k \\ &= \mu_{ki} \mu_{kj} r_{ij} x_k, \end{aligned}$$

so that the r_{ij} are normal in A. It follows that

$$r_{ij} r_{kp} = \mu_{ik} \mu_{jk} \mu_{ip} \mu_{jp} r_{kp} r_{ij} = \mu_{ik}^2 \mu_{jp}^2 r_{kp} r_{ij}, \qquad (\dagger)$$

since $\mu_{ij} = \lambda_j/\lambda_i$ for all $i,\ j$. Thus, R is a skew polynomial ring. For all $i,\ j,\ k,\ p$, let $\nu_{ijkp} = \mu_{ik}^2 \mu_{jp}^2$, so $\nu_{ijkp} \nu_{kpab} = \nu_{ijab}$, for all $i,\ j,\ k,\ p,\ a,\ b$. By Lemma 2.2, it follows that R is a twist of a polynomial ring.

For all $i,\ j$, let $c_{ij} \in B$ denote the element that twists to $r_{ij} \in A$; that is,

$$c_{ij} = X_i^\tau X_j^\tau + X_j^\tau X_i^\tau = \tau(X_i X_j + X_j X_i) \in \Bbbk^\times (X_i X_j + X_j X_i) \subset C.$$

Moreover, since B is quadratic, each Y_k is a function of the $X_iX_j + X_jX_i$ and so a function of the c_{ij}. It follows that C is the subalgebra of B generated by the c_{ij}, and so R is a twist of C. By (†), we have

$$c_{ij}c_{kp}^{\tau^2} = \nu_{ijkp}c_{kp}c_{ij}^{\tau^2},$$

for all $i,\ j,\ k,\ p$. Defining $\tau' \in \mathrm{Aut}(C)$ by $\tau'(c_{ij}) = \lambda_i^2\lambda_j^2 c_{ij}$, for all $i,\ j$ (so $\tau' = \tau^2|_C$), we find that R is a twist of C by τ'. ■

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