The One-Dimensional Line Schemes of Two Families of Potentially-Generic Quadratic Quantum $\mathbb{P}^3$s

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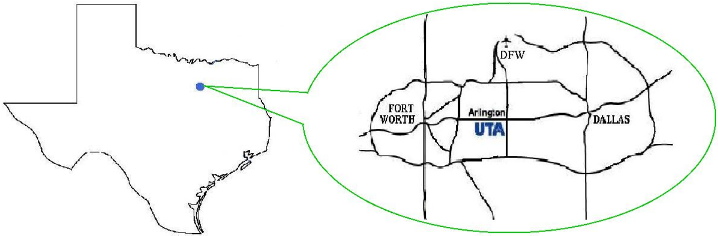

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Subgoal: identify those $\mathfrak{L}$ of dim $= 1$ where $|\mathfrak{z}| = 20$ (or $|\mathfrak{z}| < \infty$).

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Both families of algebras found by Cassidy & Vancliff several years ago & viewed as potentially-generic quadratic quantum $\mathbb{P}^3$s.

Work with Chandler

field $\mathbb{k} = \bar{\mathbb{k}}$, char$(\mathbb{k}) \neq 2$, i, $\gamma \in \mathbb{k}^\times$, $i^2 = -1$, gens $= x_1, \ldots, x_4$,

relations:

$$x_4 x_1 = i x_1 x_4, \qquad x_3^2 = x_1^2, \qquad x_3 x_1 = x_1 x_3 - x_2^2,$$
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If $\gamma(\gamma^2 - 4) \neq 0$, then $|\mathfrak{z}| = 20$ and can be grouped naturally in $\mathbb{P}^3$ into 2 sets of 2 points and 4 sets of 4 points (call latter type generic points).

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In $\mathbb{P}^3$, through each of the 16 generic points of $\mathfrak{z}$ passes exactly 6 lines of those lines parametrized by $\mathfrak{L}$ (1 line for each of the above 6 subschemes).

Work (in progress) with Tomlin

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To do: find how many lines through each point. Does this distinguish any points?

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Some References & Further Reading

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