Defining a Notion of Non-commutative Complete Intersection via Base-Point Modules

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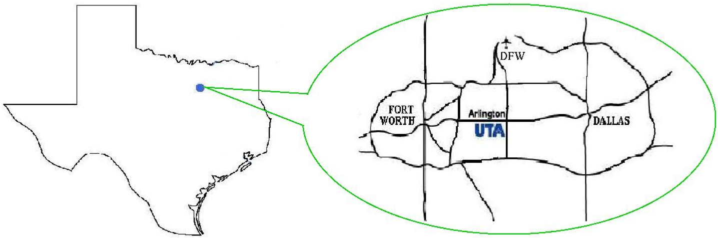

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Main Goal

... generalize the following well-known commutative result to certain non-commutative algebras satisfying good homological properties:

Theorem ([standard])

Let R denote the (commutative) polynomial ring on n variables over an algebraically closed field $\mathbb{k}$. Homogeneous elements $f_1, \ldots, f_m \in R_{>0}$ define/yield a complete intersection if one of the following equivalent conditions holds:

(a) $\{f_1, \ldots, f_m\}$ *is a regular sequence in R (i.e., for all i, f_i is not a zero divisor in $R/\langle f_1, \ldots, f_{i-1}\rangle$);*

(b) *for all $k = 1, \ldots, m$, $GKdim(R/\langle f_1, \ldots, f_k\rangle) = n - k$.*

Moreover, if $m = n$, then there exist another 2 equiv conditions equiv to (a) and (b):

(c) $\dim_{\mathbb{k}}(R/\langle f_1, \ldots, f_n\rangle) < \infty$;

(d) $\mathcal{V}(f_1, \ldots, f_n) \subset \mathbb{P}^{n-1}$ *is empty.*

Assumptions & Notation

- $\Bbbk$ = algebraically closed field.
- $A = \bigoplus_{i=0}^{\infty} A_i$ = an $\mathbb{N}$-graded, connected ($A_0 = \Bbbk$), finitely generated $\Bbbk$-algebra, generated by A_1.
- $\mathrm{GKdim}(A) = n \in \mathbb{N}$.
- A has a normalizing sequence $F = \{f_1, \ldots, f_m\} \subset A \setminus \Bbbk^{\times}$ consisting of homogeneous elements (i.e., for all i, f_i is normal in $A/\langle f_1, \ldots, f_{i-1}\rangle$).
- $I = \langle F \rangle$.

The Desired Equivalent Conditions

In order to address the notion of complete intersection in the context of A and F, we will be interested in 4 (possibly equivalent) conditions on A and F as follows:

I. F is a regular sequence in A;

II. for each $k = 1, \ldots, m$, $\text{GKdim}(A/\langle f_1, \ldots, f_k \rangle) = n - k$;
(last slide: $\text{GKdim}(A) = n$)

III. $\dim_{\Bbbk}(A/\langle F \rangle) < \infty$;

IV. $\widehat{\mathcal{V}(F)}$ is empty (to be defined below).

Base-Point Modules

In past work with T. Cassidy on skew poly rings S, he & I initially thought $\widehat{\mathcal{V}(F)}$ should be the scheme of point modules over $S/\langle F\rangle$. However, J. T. Stafford led us to realise that, even for skew poly rings, one needs to consider **both** point modules **&** fat point modules. The same is true for the algebra A in this talk....

Definition ([CV2])

A right **base-point module** over A is any 1-critical (wrt GK-dimension) graded right A-module that is generated by its homogeneous degree-zero elements & which has Hilbert series $H(t) = c/(1-t)$ for some $c \in \mathbb{N}$.

So, $c = 1 \Leftrightarrow$ point module, and $c \geq 2 \Leftrightarrow$ fat point module.

Notation: $\widehat{\mathcal{V}(F)}$ is the collection of (right) base-point modules over $A/\langle F\rangle$.

Example (See [ATV1], [CV2])

If $A =$ poly ring & $\deg(f_i) = 2$ for all i, then F is a quadric system and there is a one-to-one corresp between $\widehat{\mathcal{V}(F)}$ and the base points of the quadric system.

Proposed Definition

Definition

Let A be as above and suppose conditions I-IV are equivalent for all normalizing sequences F in A of the above form that have length n. For such an F, if the equivalent conditions I-IV hold, call $A/\langle F\rangle$ a **complete intersection**.

Are there many algebras A to which this definition applies?

Examples

- Regular skew poly rings, i.e., $S = \mathbb{k}$-algebra on generators $z_1, \ldots, z_n$ where $z_j z_i = \mu_{ij} z_i z_j$ for all $i \neq j$ such that $\mu_{ij} \in \mathbb{k}^{\times}$ & $\mu_{ij}\mu_{ji} = 1$.
- Regular graded Clifford algebras (with $\mathrm{char}(\mathbb{k}) \neq 2$).
- Regular graded skew Clifford algebras (with $\mathrm{char}(\mathbb{k}) \neq 2$).
- Coordinate ring of quantum $m \times \ell$ matrices, where $n = m\ell$.
- Homogenization of the universal enveloping algebra of any Lie algebra of dimension $n - 1 < \infty$.

Main Theorem

Initial generalization to non-commutative setting was to the case of skew poly rings with T. Cassidy in [CV1] & [CV2].

Later, I generalized it further as follows:

Theorem

Let $A = \bigoplus_{i=0}^{\infty} A_i$ *denote a connected,* $\mathbb{N}$*-graded* $\Bbbk$*-algebra that is generated by* A_1*. Suppose that A is (noetherian) Auslander-Gorenstein of finite injective dimension & satisfies the Cohen-Macaulay property, & that there exists a normalizing sequence* $\{y_1, \ldots, y_\nu\} \subset A \setminus \Bbbk^\times$ $(\ \nu < \infty\ )$ *consisting of homog elements such that* $GKdim(A/\langle y_1, \ldots, y_\nu \rangle) = 1$*. If* $GKdim(A) = n \in \mathbb{N}$*, then conditions I-IV are equivalent for any normalizing sequence* $F \subset A \setminus \Bbbk^\times$ *that consists of n homogeneous elements.*

Remarks

- Auslander-Gorenstein includes assumption A is noetherian.
- Auslander-Gorenstein & Cohen-Macaulay property allow application of results of T. Levasseur ([L]).
- Most of the result is straightforward to prove & follows from work in [CV1]; the part that needs work is III $\Leftrightarrow$ IV.

Next several slides = outline of proof of III $\Leftrightarrow$ IV.

Recall III. $\dim_{\Bbbk}(A/\langle F \rangle) < \infty$;

IV. $\widehat{\mathcal{V}(F)}$ is empty.

Outline of Proof of III $\Leftrightarrow$ IV

III. $\dim_{\Bbbk}(A/\langle F\rangle) < \infty$; IV. $\widehat{\mathcal{V}(F)} = \emptyset$

IV$'$ $\Rightarrow$ III$'$

Suppose $M = \bigoplus_{i=0}^{\infty} M_i \in \widehat{\mathcal{V}(F)}$. In particular, $M = M_0(A/\langle F\rangle)$ and $\dim_{\Bbbk}(M_0) < \infty$, while $\dim_{\Bbbk}(M) = \infty$; thus, $\dim_{\Bbbk}(A/\langle F\rangle) = \infty$.

III$'$ $\Rightarrow$ IV$'$

Let $I = \langle F\rangle$ & suppose $\dim_{\Bbbk}(A/I) = \infty$. Setting $y_0 = 0 \in A$, $\exists\ \theta \in \{0, \ldots, \nu\}$ such that $\mathrm{GKdim}(A/(I + \sum_{i=0}^{\theta} y_i A)) = 1$.

Let k denote the smallest such θ, and let $\mathcal{A} = A/(I + \sum_{i=0}^{k} y_i A)$, which is a $\Bbbk$-algebra since $\{y_1, \ldots, y_\nu\}$ is a normalizing sequence in A. By construction, $\mathrm{GKdim}(\mathcal{A}) = 1$.

We may now apply Example 5.5 of [AZ] to $\mathcal{A}$ as follows.

III$'$ $\Rightarrow$ IV$'$ (cont'd)

By [SSW], $\mathcal{A}$ is P.I. and finitely generated over its center & contains a regular homogeneous central element z of positive degree $d \in \mathbb{N}$.
Let $B = \mathcal{A}[z^{-1}]$ and $R = B_0$. The algebras $\mathcal{A}$ and B are locally finite, so $\dim_{\Bbbk}(R) < \infty$ & so R has a finite-dimensional simple module N.

Goal: build a BPM over $\mathcal{A}$ from N.

Corollary I.3.26 in [NV] $\Rightarrow B \cong R[x, x^{-1}; \sigma]$ for some $x \in B_1$, $\sigma \in \text{Aut}(R)$.

$\mathrm{III}' \Rightarrow \mathrm{IV}'$ (cont'd) $\quad (B = \mathcal{A}[z^{-1}],\ R = B_0,\ B \cong R[x, x^{-1}; \sigma])$

Still following Example 5.5 of [AZ], we can now apply Example 5.4 of [AZ], to obtain that $\mathrm{spec}(R) \cong \mathrm{proj}(\mathcal{A})$, where $\mathrm{proj}(\mathcal{A})$ is the category of fin gen graded $\mathcal{A}$-modules mod the subcat of fin gen graded torsion $\mathcal{A}$-modules.

In particular, by §7 of [ATV2], we set

$\hat{N} = \bigoplus_{i=0}^{\infty}(N \otimes_R B)_i = \bigoplus_{i=0}^{\infty}(N \otimes_R B_i)$, which is a fin gen graded $\mathcal{A}$-module, generated by $\hat{N}_0 = N \otimes \mathbb{k}$. As vector spaces, $\hat{N}_i = N \otimes \mathbb{k}x^i \quad \forall\ i$.

Goal: prove $\hat{N}$ is a BPM over $\mathcal{A}$.

It suffices to prove $\hat{N}$ is 1-critical. ($\Rightarrow \hat{N} \in \widehat{\mathcal{V}(F)}$ & so IV′ holds.)

$\mathrm{III}' \Rightarrow \mathrm{IV}'$ (cont'd) $\quad (B = \mathcal{A}[z^{-1}],\ R = B_0,\ B \cong R[x, x^{-1}; \sigma])$

By definition, $\hat{N}_i = N \otimes_R B_i$, so is an R-module.

Let $e \in \hat{N}_i^{\times}$. $\quad e = v \otimes x^i$ for some $v \in N^{\times}$.

x is normal & regular in B, so $\quad eR = (v \otimes x^i)R = v \otimes Rx^i = vR \otimes \Bbbk x^i$

$N =$ simple R-module $\Rightarrow vR = N$, so $\quad eR = N \otimes \Bbbk x^i = N \otimes_R B_i = \hat{N}_i$.

Hence, $\hat{N}_i$ is a simple R-module for all i.

Let $0 \neq M = \bigoplus_{i=r}^{\infty} M_i$ be a graded $\mathcal{A}$-submodule of $\hat{N}$, where $M_r \neq 0$.

We must prove $M_j = \hat{N}_j$ for all $j \gg 0$.

$\mathrm{III}' \Rightarrow \mathrm{IV}'$ (cont'd)

$(\ B = \mathcal{A}[z^{-1}], \ R = B_0, \ B \cong R[x, x^{-1}; \sigma] \ )$

By definition of $\hat{N}$, the central regular homog element z acts faithfully on $\hat{N}$; so z acts faithfully on M. Hence, $M_i \neq 0$ for all $i \geq r$ (as $\mathcal{A}$ generated by $\mathcal{A}_1$) &, for each $i \in \{r, \ldots, r+d-1\}$, $\{\dim_{\Bbbk}(M_{i+td})\}_{t \geq 0}$ is a nondecreasing sequence. However, $\dim_{\Bbbk}(\hat{N}_j) = \dim_{\Bbbk}(N)$ for all j, so $\dim_{\Bbbk}(M_j) = \dim_{\Bbbk}(M_{j+d})$, for all $j \gg 0$. Thus, using the faithful action of z, M_j is an R-module for all $j \gg 0$; i.e., M_j is a nonzero R-submodule of the simple R-module $\hat{N}_j$ for all $j \gg 0$; whence, $M_j = \hat{N}_j$ for all $j \gg 0$. □

Examples

Examples (revisited)

- Regular skew poly rings, i.e., $S = \Bbbk$-algebra on generators $z_1, \ldots, z_n$ where $z_j z_i = \mu_{ij} z_i z_j$ for all $i \neq j$ such that $\mu_{ij} \in \Bbbk^\times$ & $\mu_{ij}\mu_{ji} = 1$. In this case, $z_i = y_i$ from the theorem.
- Regular graded Clifford algebras (with $\text{char}(\Bbbk) \neq 2$): generators $x_1, \ldots, x_n, Y_1, \ldots, Y_n$ where $\deg(x_i) = 1$ & $\deg(Y_i) = 2$ for all i, with defining relations given by

$$x_i Y_j = Y_j x_i \quad \& \quad Y_i Y_j = Y_j Y_i \quad \text{for all } i, j, \quad \&$$
$$\forall\ i,\ j:\ x_i x_j + x_j x_i = \textstyle\sum_{k=1}^n \alpha_{ijk} Y_k, \qquad \alpha_{ijk} \in \Bbbk \text{ for all } i, j, k$$

  where each scalar matrix $M_k = (\alpha_{ijk})$ is symmetric. When the quadric system determined by the M_k is base-point free, the algebra is quadratic & regular, & $\{Y_1, \ldots, Y_n\}$ is a regular sequence, so the result applies to the algebra with $Y_i = y_i$ from the theorem.
- Regular graded skew Clifford algebras (with $\text{char}(\Bbbk) \neq 2$). Generalization of previous example.

Examples

- Coordinate ring of quantum $m \times \ell$ matrices, where $n = m\ell$.
  Results of [GL], [LS] & [L] show that such an algebra satisfies the hypotheses of the theorem.
  E.g., case $m = 2 = \ell$: generators a, b, c, d of degree one, with defining relations

$$\begin{aligned} ab &= qba, & bd &= qdb, \\ ac &= qca, & cd &= qdc, \\ bc &= cb, & ad - da &= (q - q^{-1})bc, \end{aligned}$$

  where $q \in \mathbb{k}^\times$ and $q^2 \neq 1$ [FRT]. Here, $\{b, c, d\}$ is normalizing, & factoring out $\langle b, c, d\rangle$ yields $\mathbb{k}[a]$.
- Homogenization of the universal enveloping algebra of any Lie algebra of dimension $n - 1 < \infty$.
  [LV] $\Rightarrow$ the homogenizing element z is central & regular of degree one; factoring out $\langle z\rangle$ yields the poly ring. Hence, the theorem applies.

Conclusion

Definition (revisited)

Let A be as in the main theorem. If F is a normalizing sequence in A that consists of n homogeneous elements of positive degree, call $A/\langle F \rangle$ a **complete intersection** if the equivalent conditions I-IV hold.

Remarks

- All the above examples are Auslander-regular and Artin-Schelter regular. So, for those of us who view a regular algebra as being a non-commutative analogue of the poly ring, this is perhaps further support of that viewpoint.
- Since, in general, not all algebras have normalizing elements (or enough of them), other notions of complete intersection for non-commutative algebras are being investigated by researchers, such as the recent work in [KKZ] that uses a more homological approach than that used here.

Question

Can the result (or a modified version thereof) be extended to AS-regular algebras without the assumption of Auslander-Gorenstein?

Question

Let $I_k = \langle f_1, \ldots, f_k \rangle$ for all $k \leq m \leq n$. If A is commutative, then, for each k, $\widehat{\mathcal{V}(I_k)}$ is a (projective) scheme, and so has a dimension. In particular, if A is the polynomial ring, then F is regular if and only if $\dim(\widehat{\mathcal{V}(I_k)}) = n - k - 1$, for all $k \leq m$. However, if A is not commutative, is there an analogous statement and under what hypotheses on A could it hold?

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