GENERALIZATIONS OF GRADED CLIFFORD ALGEBRAS AND OF COMPLETE INTERSECTIONS

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Abstract. For decades, the study of graded Clifford algebras has provided a theory where commutative algebraic geometry has dictated the algebraic and homological behavior of a non-commutative algebra. In particular, it is well known that a graded Clifford algebra, C, is quadratic and regular if and only if a certain quadric system associated to C is base-point free. In this article, we introduce a generalization of a graded Clifford algebra, namely a *graded skew Clifford algebra*, and to such an algebra we associate a notion of quadric system in the spirit of the non-commutative algebraic geometry of Artin, Tate and Van den Bergh. We prove that a graded skew Clifford algebra is quadratic and regular if and only if its associated quadric system is normalizing and base-point free. To prove our results, we extend the notions of complete intersection, base-point free, quadratic form and symmetric matrix to the non-commutative setting.

We use our results to produce several families of quadratic regular skew Clifford algebras of global dimension four that are not twists of graded Clifford algebras. Many of our examples have a 1-dimensional line scheme and a point scheme that consists of exactly twenty distinct points, and so are candidates for generic regular algebras of global dimension four.

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Introduction

For decades, the study of graded Clifford algebras has been an area where commutative algebraic geometry has dictated the algebraic and homological behavior of a non-commutative algebra. Indeed, associated to a finitely generated graded Clifford algebra, C, is a system of quadrics in a finite-dimensional projective space, and, by [4, 7], C is quadratic and regular if and only if the associated quadric system is base-point free (Theorem 1.1 gives the precise statement).

In this article, we introduce an algebra that generalizes the notion of a graded Clifford algebra, and we call such an algebra a *graded skew Clifford algebra* (Definition 1.12). Using the theory of non-commutative algebraic geometry developed in [2, 3], we associate geometric data to a graded skew Clifford algebra, analogous to that associated to a graded Clifford algebra. In particular, in the non-commutative setting, we define a quadric, a quadric system and what it means for a quadric system to be base-point free (Definitions 1.5, 1.11 and 1.7), and our definitions extend the analogous concepts from commutative algebra. Although, in general, such a non-commutative quadric might contain only finitely many points, we prove in Theorem 4.2 that a graded skew Clifford algebra is quadratic and regular if and only if the associated quadric system is normalizing and base-point free.

In our results, a key part is played by quadratic forms in a skew polynomial ring S on n generators, and there is a certain $n \times n$ matrix that may be associated to such a quadratic form. We call such a matrix a μ-symmetric matrix, where μ is an $n \times n$ scalar matrix depending on S (Definition 1.2). Moreover, there is a one-to-one correspondence between quadratic forms in S and μ-symmetric matrices, and they interact in much the same way as do symmetric matrices with their associated quadratic forms in a polynomial ring. Symmetric matrices (respectively, skew-symmetric matrices) are μ-symmetric matrices for certain choices of μ, and so μ-symmetric matrices generalize the concepts of symmetric, and skew-symmetric, matrix.

Given a graded skew Clifford algebra A, determining the influence of its associated quadric system necessitates consideration of a non-commutative analog of the concept of complete intersection. To this end, we invoke many results of T. Levasseur from [8], and we develop a notion of complete intersection for the non-commutative algebraic geometry associated to skew polynomial rings. Our main result, Theorem 4.2, follows from using our non-commutative analog of complete intersection in conjunction with results of Shelton and Tingey in [9] that use regular normalizing sequences of elements in regular algebras to construct new regular algebras.

A consequence of Theorem 4.2 is that a regular graded skew Clifford algebra may be constructed by simply finding a normalizing quadric system that is base-point free. Producing a normalizing quadric system entails the entries of the matrix μ satisfying some algebraic conditions; although this is a closed property, it is not as restrictive as in the commutative setting where all the entries of μ equal one. Once a normalizing quadric system is found, the property of being base-point free is an open condition.

Given the closed condition satisfied by the entries of μ, regular graded skew Clifford algebras are not, in general, flat deformations of regular graded Clifford algebras, and this agrees with work of C. Ingalls which proves that, in general, flat deformations of regular graded Clifford algebras are still regular graded Clifford algebras. Nevertheless, the family of regular graded skew Clifford algebras promises to contribute to resolving the open problem of classifying all quadratic regular algebras of global dimension four, so-called quantum $\mathbb{P}^3$s. Indeed, it is likely that regular graded skew Clifford algebras of global dimension four yield many components of the scheme of quadratic algebras on four generators with six relations, including some such algebras that are considered generic quantum $\mathbb{P}^3$s. This claim is supported by the many examples of quadratic regular skew Clifford algebras of global dimension four that have 1-dimensional line schemes and point schemes consisting of exactly twenty distinct points that are presented in the last section, and which are candidates for generic quantum $\mathbb{P}^3$s. Although various researchers have found examples of quantum $\mathbb{P}^3$s having finite point schemes and 1-dimensional line schemes (c.f., [5, 9, 13, 16]), before this article, only one example had exactly twenty distinct points *and* a 1-dimensional line scheme, and that algebra appeared in [9] in 2001; however, even that algebra is a graded skew Clifford algebra.

The article is outlined as follows. Section 1 presents known results for graded Clifford algebras and skew polynomial rings, and introduces graded skew Clifford algebras and their associated (non-commutative) quadric system. In Section 2, we focus on certain noetherian Auslander-Gorenstein algebras, and prove some results concerning the Gelfand-Kirillov dimension of quotients of such an algebra by regular normalizing sequences. We connect the results of Section 2 with the geometry of skew polynomial rings in Section 3, and prove, in Corollary 3.6, that an analog of complete intersection occurs for skew polynomial rings. In Section 4, we prove our main result (Theorem 4.2) that a graded skew Clifford algebra is quadratic and regular if and only if its associated quadric system is normalizing and base-point free. We use this result to prove that all graded regular algebras of global dimension two are graded skew Clifford algebras. We also prove, in Theorem 4.6, that if a quadric system $\mathfrak{Q}$ is normalizing and base-point free, then a "hyperplane slice" of the quadric system corresponds to a regular graded skew Clifford subalgebra of the regular graded skew Clifford algebra corresponding to $\mathfrak{Q}$. Another result in Section 4 gives necessary and sufficient conditions for a regular graded skew Clifford algebra to be a twist by an automorphism of a regular graded Clifford algebra. Some examples of regular graded skew Clifford algebras of global dimension four and their associated quadric systems are presented in Section 5. Our examples are not isomorphic to, nor twists of, graded Clifford algebras, and most of our examples have finite point schemes and 1-dimensional line schemes, including some that have exactly twenty distinct points.

1. Skew Clifford Algebras and Skew Polynomial Rings

In this section, we introduce the algebras to be discussed in the paper, namely graded skew Clifford algebras, and associate to them some geometric data.

Throughout the article, $\Bbbk$ denotes an algebraically closed field such that $\text{char}(\Bbbk) \neq 2$, and $M_n(\Bbbk)$ denotes the vector space of $n \times n$ matrices with entries in $\Bbbk$. For a graded $\Bbbk$-algebra B, the span of the homogeneous elements in B of degree i will be denoted B_i. The notation $T(V)$ will denote the tensor algebra on the vector space V, and, if R is any ring or vector space, then $R^\times$ will denote the nonzero elements in R.

1.1. **Clifford Algebras.**

Let $M_1, \ldots, M_n \in M_n(\Bbbk)$ denote symmetric matrices. By definition (c.f., [7]), a graded Clifford algebra is the $\Bbbk$-algebra C on degree-one generators $x_1, \ldots, x_n$ and on degree-two generators $y_1, \ldots, y_n$ with defining relations given by

(a) $x_i x_j + x_j x_i = \sum_{k=1}^{n} (M_k)_{ij}\, y_k$ for all $i, j = 1, \ldots, n$, and

(b) y_k central for all $k = 1, \ldots, n$.

It is possible for all the y_k to belong to $(C_1)^2$, and this happens if and only if $M_1, \ldots, M_n$ are linearly independent. However, even in this case, C need not be a quadratic algebra, as the centrality of all the y_k might not follow from the degree-two relations.

There is a geometry that may be associated to C via the matrices M_k as follows. To each M_k, associate a quadratic form $q_k \in \Bbbk[z_1, \ldots, z_n]$ given by $q_k = z^T M_k z$, where

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \qquad \text{and} \qquad z^T = \begin{bmatrix} z_1 & \ldots & z_n \end{bmatrix}.$$

If we view $\{z_1, \ldots, z_n\}$ as a dual basis in $(C_1)^*$ to $\{x_1, \ldots, x_n\}$, then the polynomial algebra $\Bbbk[z_1, \ldots, z_n]$ is the Koszul dual of the quadratic algebra $C/\langle y_1, \ldots, y_n\rangle$; that is, let $W \subset T(C_1)_2$ denote the preimage in $T(C_1)$ of the relations of $C/\langle y_1, \ldots, y_n\rangle$ and let $W^\perp \subset (C_1)^* \otimes_\Bbbk (C_1)^*$ denote the homogeneous elements of degree two in $T((C_1)^*)$ that vanish on the elements of W, then $\mathbb{k}[z_1, \ldots, z_n] = T((C_1)^*)/\langle W^\perp \rangle$. With this viewpoint, the zero locus $\mathcal{V}(q_k)$ of each q_k is a quadric in $\mathbb{P}(C_1)$. A base point of the quadric system $\mathcal{V}(q_1), \ldots, \mathcal{V}(q_n)$ is a point in $\bigcap_{k=1}^{n} \mathcal{V}(q_k)$; if no such point exists, then the quadric system is said to be base-point free.

Theorem 1.1. [4, 7] *The graded Clifford algebra C is quadratic, Auslander-regular of global dimension n and satisfies the Cohen-Macaulay property with Hilbert series $1/(1-t)^n$ if and only if the associated quadric system is base-point free; in this case, C is Artin-Schelter regular and is a noetherian domain.*

We intend to generalize the notion of a graded Clifford algebra, defining a graded *skew Clifford algebra* in Definition 1.12, and then prove a result analogous to Theorem 1.1 for graded skew Clifford algebras. To do so will entail generalizing the notions of quadric and base-point free to the non-commutative setting via skew polynomial rings.

1.2. **Skew Polynomial Rings.**

Throughout the article, for $\{i, j\} \subset \{1, \ldots, n\}$, let $\mu_{ij} \in \mathbb{k}^\times$ satisfy the following property:

$$\mu_{ij}\mu_{ji} = 1 \quad \text{for all } i \neq j.$$

We write $\mu = (\mu_{ij}) \in M_n(\mathbb{k})$.

Definition 1.2. A matrix $M \in M_n(\mathbb{k})$ will be called *μ-symmetric* if $M_{ij} = \mu_{ij} M_{ji}$ for all $i, j = 1, \ldots, n$.

If $\mu_{ij} = 1$ for all i, j, then a μ-symmetric matrix is a symmetric matrix, whereas if $\mu_{ij} = -1$ for all i, j, then a μ-symmetric matrix is a skew-symmetric matrix. It will be convenient for us to assume that $\mu_{ii} = 1$ for all i, and henceforth we assume that is the case.

Let S denote the $\mathbb{k}$-algebra on $z_1, \ldots, z_n$ with defining relations given by $z_j z_i = \mu_{ij} z_i z_j$ for all $i, j = 1, \ldots, n$, so that S is a skew polynomial ring. In the commutative case (where all the $\mu_{ij} = 1$), one may associate a quadratic form to a symmetric matrix $M \in M_n(\mathbb{k})$ in the way outlined in Subsection 1.1 for the M_k and q_k. It should be noted that in doing so, one first forms $z^T M z$ in $T(S_1)$, after which one uses the relations of the polynomial algebra to rewrite the quadratic form more simply. We now propose a similar procedure if $\mu_{ij} \neq 1$ for some $i, j = 1, \ldots, n$.

Let $M \in M_n(\Bbbk)$ be a μ-symmetric matrix, and let $\tilde{q} \in T(S_1)$ be defined by $z^T M z$, where z and z^T are column and row matrices as in Subsection 1.1. Let $q \in S$ denote the image of $\tilde{q}$ and we call q a quadratic form. Moreover, given $q = \sum_{i \le j} \alpha_{ij} z_i z_j \in S_2$, one may associate to q a μ-symmetric matrix $M = (M_{ij})$ as follows: set $M_{ii} = \alpha_{ii}$ for all i, and, for $i < j$, set $M_{ij} = \frac{\alpha_{ij}}{2}$ and $M_{ji} = \mu_{ji} M_{ij}$. Moreover,

$$q = \sum_{i,j} M_{ij} z_i z_j = \sum_{i \le j} M_{ij} z_i z_j + \sum_{i<j} M_{ji} \mu_{ij} z_i z_j = \sum_{i<j} 2M_{ij} z_i z_j + \sum_i M_{ii} z_i^2, \qquad (*)$$

so that $q \neq 0$ if and only if $M \neq 0$.

Lemma 1.3. *If $q_1, \ldots, q_n \in S_2$ and if M_k denotes the μ-symmetric matrix associated to each q_k, then $\{q_k\}_{k=1}^n$ is linearly independent in S if and only if $\{M_k\}_{k=1}^n$ is linearly independent.*

Proof. By $(*)$, if $\alpha_1, \ldots, \alpha_n \in \Bbbk$, then

$$\sum_{k=1}^n \alpha_k q_k = \sum_{k=1}^n \alpha_k \Big(\sum_{i<j} 2(M_k)_{ij} z_i z_j + \sum_i (M_k)_{ii} z_i^2 \Big).$$

Given the defining relations of S, it follows that $\sum_{k=1}^n \alpha_k (M_k)_{ij} = 0$ for all $i \le j$ if and only if $\sum_{k=1}^n \alpha_k q_k = 0$ in S, which completes the proof. ■

Lemma 1.3 reinforces the fact that the association between μ-symmetric matrices and quadratic forms in S mirrors the standard association between symmetric matrices and quadratic forms in polynomial rings.

In order to associate a quadric to any $q \in S_2$, we first set $U \subset T(S_1)_2$ to be the defining relations of S, and write $\mathcal{V}(U)$ for the zero locus in $\mathbb{P}((S_1)^*) \times \mathbb{P}((S_1)^*)$ of U.

Remark 1.4. In the language of [2] (and Definition 3.1), $\mathcal{V}(U)$ parametrizes the point modules over the skew polynomial ring S. In general, since $Sz_k = z_k S$ for all k, a point module over S

that is annihilated by any z_k corresponds to a point module over $S/\langle z_k\rangle$ and so, by an inductive argument, it follows that $1 \leq \dim(\mathcal{V}(U)) \leq n-1$. Moreover, in the commutative setting, where $\mu_{ij} = 1$ for all $i,\ j$, $\mathcal{V}(U)$ is the graph of the identity map on $\mathbb{P}((S_1)^*)$, so that $\mathcal{V}(U)$ may be identified with $\mathbb{P}((S_1)^*)$ in this case.

Definition 1.5. For any $q \in S_2$, we call the zero locus of q in $\mathcal{V}(U)$ the *quadric associated to* q, and denote it by $\mathcal{V}_{_U}(q)$; in other words, $\mathcal{V}_{_U}(q) = \mathcal{V}(\Bbbk\hat{q} + U) = \mathcal{V}(\hat{q}) \cap \mathcal{V}(U)$, where $\hat{q}$ is any lift of q to $T(S_1)_2$. The span of elements $q_1, \ldots, q_m$ in S_2 will be called the *quadric system* associated to $q_1, \ldots, q_m$.

A more general notion of (non-commutative) quadric has been proposed independently by M. Van den Bergh in [17]; the two notions are related, but the one given in Definition 1.5 suffices for our goals in this article.

Lemma 1.6. *If $n \geq 2$, then the quadric $\mathcal{V}_{_U}(q)$ is nonempty for all $q \in S_2^{\times}$ and has codimension one in $\mathcal{V}(U)$.*

Proof. Let $\phi : \mathbb{P}((S_1)^*) \times \mathbb{P}((S_1)^*) \to \mathbb{P}((S_1 \otimes_{\Bbbk} S_1)^*)$ denote the Segre embedding, where $\phi(a, b) = ab^T$ (viewing a and b as column vectors). Correspondingly, any nonzero element $q \in T(S_1)_2$ maps to a nonzero element of $T(S_1 \otimes_{\Bbbk} S_1)_1$. Since $Sz_k = z_kS$ for all k, both $\mathcal{V}(U)$ and $\phi(\mathcal{V}(U))$ have dimension at least one. Moreover, if $q \in S_2^{\times}$, and if $\hat{q}$ is any lift of q to $T(S)_2$, then $\phi(\mathcal{V}_{_U}(q)) = \phi(\mathcal{V}(U)) \cap \phi(\mathcal{V}(\hat{q}))$, so, by intersection theory in $\mathbb{P}((S_1 \otimes_{\Bbbk} S_1)^*)$, we have

$$\begin{aligned} \dim(\mathcal{V}_{_U}(q)) &= \dim(\phi(\mathcal{V}(U))) + \dim(\phi(\mathcal{V}(\hat{q}))) - \dim(\mathbb{P}((S_1 \otimes_{\Bbbk} S_1)^*)) \\ &= \dim(\phi(\mathcal{V}(U))) + (n^2 - 2) - (n^2 - 1) \\ &= \dim(\phi(\mathcal{V}(U))) - 1 \\ &\geq 0. \end{aligned}$$

The result follows. ■

Definition 1.7. We call a point $(a,\ b) \in \mathcal{V}(U)$ a *base point* of the quadric system associated to $q_1, \ldots, q_m \in S_2$ if $(a,\ b) \in \mathcal{V}_U(q_k)$ for all $k = 1, \ldots, m$. We say such a quadric system is *base-point free* if $\bigcap_{k=1}^{m} \mathcal{V}_U(q_k)$ is empty.

Remark 1.8.

(a) Definitions 1.5 and 1.7 reduce to the usual definitions in the commutative setting (where $\mu_{ij} = 1$ for all i, j), providing $\mathcal{V}(U)$ is identified with $\mathbb{P}((S_1)^*)$ as described in Remark 1.4.

(b) If $n = 4$, and if $\{\mu_{ij}\}_{i<j}$ is algebraically independent, then $\mathcal{V}(U)$ has dimension one, so a quadric need not have codimension one relative to the dimension of $\mathbb{P}((S_1)^*)$.

(c) In the language of [2], the quadric system $\mathcal{V}_U(q_1), \ldots, \mathcal{V}_U(q_m)$ being base-point free is equivalent to the algebra $S/\langle q_1, \ldots, q_m\rangle$ having no truncated point modules of length three.

In order for our non-commutative quadric system to be useful in our analog of Theorem 1.1 (namely, Theorem 4.2), the quadratic forms should satisfy some additional properties. To that end, we now remind the reader of some standard terminology.

Definition 1.9. Let B denote a $\Bbbk$-algebra. The following terms are standard.

(a) An element $b \in B$ is said to be a *normal* element if $Bb = bB$. If $b_1, \ldots, b_m \in B$ and if the image of b_{k+1} in $B/\langle b_1, \ldots, b_k\rangle$ is normal for all $k = 1, \ldots, m-1$, then $\{b_1, \ldots, b_m\}$ is called a *normalizing sequence* in B.

(b) A normal element $b \in B$ is said to be *regular* if the only solution $x \in B$ to the equation $xb = 0$ is $x = 0$. A normalizing sequence $\{b_1, \ldots, b_m\}$ in B is said to be *regular* if b_{k+1} is regular in $B/\langle b_1, \ldots, b_k\rangle$ for all $k = 1, \ldots, m-1$.

(c) If B is $\mathbb{N}$-graded and if $B_0 = \Bbbk$, then a homogeneous element $b \in B$ is said to be *1-regular* if the only solutions $x, y \in B_1$ to the equations $xb = 0$ and $by = 0$ are $x = 0$ and $y = 0$.

Remark 1.10. Normality and 1-regularity of elements have influence on elements in certain Koszul duals as follows ([9]). Let B denote a quadratic $\Bbbk$-algebra such that $B = T(V)/\langle R\rangle$, where V is a finite-dimensional vector space over $\Bbbk$ and R is a subspace of $V \otimes_{\Bbbk} V$. Let $b \in B_2^{\times}$, and let $\hat{b}$ denote any lift of b to $T(V)_2$. There exists $b^{\star} \in (V^* \otimes_{\Bbbk} V^*)/(R \oplus \Bbbk\hat{b})^{\perp}$ such that $R^{\perp} = (R \oplus \Bbbk\hat{b})^{\perp} \oplus \Bbbk\hat{b^{\star}}$, where $\hat{b^{\star}}$ is any lift of $b^{\star}$ to $V^* \otimes_{\Bbbk} V^*$. Since $(V^* \otimes_{\Bbbk} V^*)/(R \oplus \Bbbk\hat{b})^{\perp}$ is a subspace of $T(V^*)/\langle(R \oplus \Bbbk\hat{b})^{\perp}\rangle$, we also use $\hat{b^{\star}}$ for the image of $\hat{b^{\star}}$ in $T(V^*)/\langle(R \oplus \Bbbk\hat{b})^{\perp}\rangle$, where the latter algebra is the Koszul dual of $B/\langle b\rangle$. We note that $b^{\star}$ is well defined up to a scalar multiple. By [9, Lemma 1.3], b is normal in B if and only if $b^{\star}$ is 1-regular in $T(V^*)/\langle(R \oplus \Bbbk\hat{b})^{\perp}\rangle$.

Definition 1.11. If a quadric system is given by a normalizing sequence in S, then it is called a *normalizing quadric system*.

The reader should note that if $\{q_1, \ldots, q_m\} \subset S_2$ is not a normalizing sequence in S, then, conceivably, there could exist a normalizing sequence $\{q'_1, \ldots, q'_m\} \subset S_2$ such that $\sum_{k=1}^{m} \Bbbk q'_k = \sum_{k=1}^{m} \Bbbk q_k$, in which case the quadric system associated to $\{q_1, \ldots, q_m\}$ is normalizing.

1.3. **Skew Clifford Algebras.**

As in the preceding subsection, let $\mu \in M_n(\Bbbk)$, with $\mu_{ii} = 1$ for all i, and let S be the corresponding skew polynomial ring on $z_1, \ldots, z_n$. Let $M_1, \ldots, M_n$ denote μ-symmetric matrices. We write $\tilde{q}_k \in T(S_1)_2$ for $z^T M_k z$ and use q_k for the image of $\tilde{q}_k$ in S. Let U denote the span of the defining relations of S in $T(S_1)_2$.

Definition 1.12. A *graded skew Clifford algebra* $A = A(\mu, M_1, \ldots, M_n)$ associated to μ and M_1, $\ldots$, M_n is a graded $\Bbbk$-algebra on degree-one generators $x_1, \ldots, x_n$ and on degree-two generators $y_1, \ldots, y_n$ with defining relations given by:

(a) $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^{n} (M_k)_{ij} y_k$ for all $i, j = 1, \ldots, n$, and
(b) the existence of a normalizing sequence $\{r_1, \ldots, r_n\}$ that spans $\Bbbk y_1 + \cdots + \Bbbk y_n$.

If $\mu_{ij} = 1$ for all $i, j = 1, \ldots, n$, and if the r_i are central for all $i = 1, \ldots, n$, then a graded skew Clifford algebra $A(\mu, M_1, \ldots, M_n)$ is a graded Clifford algebra in the usual sense (Subsection 1.1).

Although our definition of a graded skew Clifford algebra A does not, in general, determine A uniquely for any given $\mu, M_1, \ldots, M_n$, we will find that, in applications, A will be quadratic, so that the relations in Definition 1.12(b) will be determined by those in Definition 1.12(a).

In order to associate geometry to a graded skew Clifford algebra, we establish some notation similar to that in Subsection 1.1 for a graded Clifford algebra. With S and U as above, write $V = A_1$, view S_1 as V^*, and view $\{z_1, \ldots, z_n\}$ as the dual basis in V^* to $\{x_1, \ldots, x_n\}$. Let $V' = V \oplus V''$, where $V'' = \bigoplus_{k=1}^{n} \Bbbk y_k$.

Let $W' \subset V \otimes_{\Bbbk} V$ be the subspace

$$W' = \sum_{1 \leq i \leq j \leq n} \Bbbk(x_i \otimes x_j + \mu_{ij} x_j \otimes x_i) \subset V \otimes_{\Bbbk} V,$$

so that $T(V')/\langle W' + V'' \rangle$ is the quadratic algebra $T(V)/\langle W' \rangle$, which is the Koszul dual of S, since $(W')^{\perp} = U \subset T(S_1)_2$.

We set $Y = \sum_{k=1}^{n} M_k y_k$, so that the entries Y_{ij} of Y satisfy $Y_{ij} = \mu_{ij} Y_{ji}$ for all $i, j = 1, \ldots, n$. With this notation, $x_i x_j + \mu_{ij} x_j x_i = Y_{ij}$ in A for all $i, j = 1, \ldots, n$.

Lemma 1.13. *With the above notation, the following are equivalent:*

(a) $y_i \in (A_1)^2$ *for all* $i = 1, \ldots, n$

(b) $\dim \left(\sum_{i,j=1}^{n} \Bbbk Y_{ij} \right) = n$

(c) $M_1, \ldots, M_n$ *are linearly independent.*

Proof. Since $\sum_{i,j=1}^{n} \Bbbk Y_{ij} \subset V''$ and $\dim(V'') = n$, it follows that if (b) holds, then (a) holds. Conversely, if $\dim \left(\sum_{i,j=1}^{n} \Bbbk Y_{ij} \right) < n$, then there exists k such that

$$y_k \notin \sum_{i,j=1}^{n} \Bbbk Y_{ij} = \sum_{i,j=1}^{n} \Bbbk(x_i x_j + \mu_{ij} x_j x_i).$$

However, the only relations of A that relate the y_i to elements of $(A_1)^2$ are given by Definition 1.12(a), so it follows that $y_k \notin (A_1)^2$. Thus, if (a) holds, then (b) holds.

(c) fails if and only if there exist $\alpha_1, \ldots, \alpha_n \in \Bbbk$, not all zero, such that $\sum_{k=1}^{n} \alpha_k M_k = 0$. This is equivalent to $Y|_\alpha = 0$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{P}((V'')^*)$. However, $Y|_\alpha = 0$ for some $\alpha \in \mathbb{P}((V'')^*)$ if and only if the zero locus of $\sum_{i,j=1}^{n} \Bbbk Y_{ij}$ in $\mathbb{P}((V'')^*)$ is nonempty, which happens if and only if $\dim\left(\sum_{i,j=1}^{n} \Bbbk Y_{ij}\right) \leq n-1$. Hence, (b) is equivalent to (c). ■

Henceforth, unless otherwise stated, we assume $M_1, \ldots, M_n$ are linearly independent, so that $y_i \in (A_1)^2$ for all $i = 1, \ldots, n$. In particular, $T(V) \twoheadrightarrow A$, so the relations in Definition 1.12(a) have a preimage W in $T(V)_2$; that is, $W = (V \otimes_{\Bbbk} V) \cap J$, where J is the span in $T(V')_2$ of the relations given by Definition 1.12(a).

Lemma 1.14. *The graded skew Clifford algebra $A(\mu, M_1, \ldots, M_n)$ is a homomorphic image of the quadratic algebra $T(V)/\langle W \rangle$, and $\dim(W) = \binom{n}{2}$.*

Proof. We have that $T(V) \twoheadrightarrow A$, and since W is contained in the span of the defining relations of A, the first statement follows.

The algebra A has $\binom{n}{2} + n$ independent relations of the type in Definition 1.12(a), and n of these relations are used to write A as a quotient of $T(V)$ (i.e., to write each $y_i \in (A_1)^2$). Hence, $\dim(W) = \binom{n}{2}$. ■

Analogous to the setting of graded Clifford algebras, the graded skew Clifford algebra A is quadratic if and only if $A = T(V)/\langle W \rangle$. However, even if A is not quadratic, we still have the following maps of quadratic algebras:

$$\frac{T(V)}{\langle W \rangle} \twoheadrightarrow \frac{T(V)}{\langle W' \rangle} = \frac{T(V)}{\langle W + V'' \rangle}$$

and

$$S = \frac{T(V^*)}{\langle U \rangle} \twoheadrightarrow \frac{T(V^*)}{\langle W^\perp \rangle} = \frac{S}{\langle r_n^\star, \ldots, r_1^\star \rangle},$$

where the notation $r_k^\star$ is from Remark 1.10 and Subsection 1.2.

If A is a graded Clifford algebra (so all the $\mu_{ij} = 1$), the algebra $T(V^*)/\langle U \rangle$ is simply the commutative polynomial ring on $z_1, \ldots, z_n$, and the algebra $T(V^*)/\langle W^\perp \rangle$ is the quotient of this polynomial ring by $\langle q_1, \ldots, q_n \rangle$. So the case of graded Clifford algebras goes hand-in-hand with commutative algebraic geometry, by associating Koszul duals to the appropriate algebras. In particular, in this case, W' is the subspace, $\mathrm{Symm}(V \otimes_{\Bbbk} V)$, consisting of the symmetric elements in $V \otimes_{\Bbbk} V$.

Moreover, if all the $\mu_{ij} = 1$ and if the quadric system is base-point free, then Theorem 1.1 implies that the Koszul dual, $T(V)/\langle W \rangle$, of $T(V^*)/\langle W^\perp \rangle$ is the associated graded Clifford algebra A; in particular, the fact that the y_i are central in A is a consequence of the quadric system being base-point free.

If we consider $z^T Y z = \sum_{i,j,k} z_i (M_k)_{ij} y_k z_j$ as an element of $T(S_1) \otimes_{\Bbbk} T(V'')$, then we may evaluate $z^T Y z$ on $V \otimes_{\Bbbk} V$ as follows:

$$(z^T Y z)(x_s \otimes x_t) = \sum_{i,j,k} z_i(x_s)(M_k)_{ij} y_k z_j(x_t) = \sum_k (M_k)_{st} y_k = Y_{st},$$

for any $s, t = 1, \ldots, n$.

Lemma 1.15. *If $w \in W'$, then $w \in W$ if and only if $(z^T Y z)(w) = 0$.*

Proof. If $\alpha_{ij} \in \Bbbk$, for $i, j = 1, \ldots, n$, then

$$(z^T Y z) \left(\sum_{1 \le i \le j \le n} \alpha_{ij} (x_i \otimes x_j + \mu_{ij} x_j \otimes x_i) \right) = \sum_{1 \le i \le j \le n} \alpha_{ij} (Y_{ij} + \mu_{ij} Y_{ji}) = 2 \sum_{1 \le i \le j \le n} \alpha_{ij} Y_{ij}.$$

Since $\sum_{1 \le i \le j \le n} \alpha_{ij} Y_{ij} = 0$ if and only if $\sum_{1 \le i \le j \le n} \alpha_{ij} (x_i x_j + \mu_{ij} x_j x_i) = 0$ in A, the result follows. ■

Corollary 1.16. *For each $k = 1, \ldots, n$, $\tilde{q}_k \in W^\perp$, and $\{q_k\}_{k=1}^n$ is a basis for $W^\perp / U \subset S$.*

Proof. Let $e_k = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{P}((V'')^*)$, where the 1 appears in the kth coordinate. For each $k = 1, \ldots, n$, $M_k = Y|_{e_k}$, so that $\tilde{q}_k = z^T M_k z \in W^\perp$ by Lemma 1.15. Since $M_1, \ldots, M_n$ are linearly independent by our ongoing assumption, Lemma 1.3 implies that

$q_1, \ldots, q_n$ are linearly independent in S, and hence in $W^\perp/U$. Moreover, $\dim\left(W^\perp/U\right) = n^2 - 2\binom{n}{2} = n$, so $\{q_k\}_{k=1}^n$ spans $W^\perp/U$. ■

Remark 1.17. Since $S/\langle r_n^\star, \ldots, r_1^\star\rangle = T(S_1)/\langle W^\perp\rangle$, it follows from Corollary 1.16 that $\sum_{k=1}^n \Bbbk r_k^\star = \sum_{k=1}^n \Bbbk q_k$.

Corollary 1.18. *Suppose $a, b \in V$. If $a \otimes b \in W'$, then $ab = 0$ in A if and only if q_k vanishes on the point $(a,\ b) \in \mathcal{V}(U) \subset \mathbb{P}(V) \times \mathbb{P}(V)$ for all $k = 1, \ldots, n$.*

Proof. The result follows immediately from Lemma 1.15 and Corollary 1.16. ■

By Definition 1.7, this result implies that if $a, b \in V$ and if $a \otimes b \in W'$, then $ab = 0$ in A if and only $(a,\ b)$ is a base point of the quadric system $\mathcal{V}_U(q_1), \ldots, \mathcal{V}_U(q_n)$. Theorem 4.2 will make the connection between base points of the quadric system and zero divisors in A even stronger.

2. Regular Sequences

In this section, we will prove results concerning regular normalizing sequences of elements in a $\Bbbk$-algebra satisfying certain natural conditions. In Section 3, we will connect these results to geometric data in order to extend the notion of complete intersection from the commutative setting to that of skew polynomial rings.

We will use $H_B(t)$ to denote the Hilbert series of a $\mathbb{Z}$-graded locally finite $\Bbbk$-algebra B, and $\text{GKdim}(M)$ to denote the Gelfand-Kirillov dimension (GK-dimension) of a module M.

Let S denote the commutative polynomial ring on n variables of degree one, and let $f_1, \ldots, f_m$ be homogeneous elements of S of positive degree. For each $k = 1, \ldots, m$, let $S(k) = S/\langle f_1, \ldots, f_k\rangle$. If $\{f_1, \ldots, f_m\}$ is a regular sequence, then $\text{GKdim}(S(k)) = n - k$, for all $k = 1, \ldots, m$. In $\mathbb{P}^{n-1}$, this corresponds geometrically to $\dim\left(\bigcap_{i=1}^k \mathcal{V}(f_i)\right) = n - k$, for all

$k = 1, \ldots, m$, where $\mathcal{V}(f_i)$ denotes the zero locus of f_i in $\mathbb{P}^{n-1}$ for each i. In this case, the geometric setting and the algebra $S(m)$ are called a complete intersection.

In particular, a quadric system $\mathcal{V}(q_1), \ldots, \mathcal{V}(q_n)$ associated to a graded Clifford algebra (as in Subsection 1.1) is base-point free if and only if it is a complete intersection. In the notation of Section 1, this is equivalent to $\{q_1, \ldots, q_n\}$ being a regular sequence in the polynomial ring $T(V^*)/\langle U\rangle$. We aim to extend this result to normalizing sequences in the case where $T(V^*)/\langle U\rangle$ is a skew polynomial ring.

Lemma 2.1. *If B is a locally finite, $\mathbb{Z}$-graded $\mathbb{k}$-algebra, and if $q \in B_d^\times$ is a regular, normal element of B that is not a unit, then the Hilbert series of $B/\langle q\rangle$ is $(1-t^d)H_B(t)$.*

Proof. The result clearly holds if $d \neq 0$. If $d = 0$, then, since B is locally finite, there exists a minimal $n \in \mathbb{N}$ such that $q^n = \sum_{i=0}^{n-1} \alpha_i q^i$, where $\alpha_i \in \mathbb{k}$ for all i. If $\alpha_0 = 0$, then q is not regular, and if $\alpha_0 \neq 0$, then q is a unit; hence, $d \neq 0$. ■

Corollary 2.2. *Suppose B is a $\mathbb{Z}$-graded $\mathbb{k}$-algebra with Hilbert series $1/(1-t)^n$ for some $n \in \mathbb{N} \setminus \{0\}$. If $\{f_1, \ldots, f_n\} \subset B_2$ is a regular normalizing sequence in B, then the Hilbert series of $B/\langle f_1, \ldots, f_n\rangle$ is $(1+t)^n$.*

Proof. The Hilbert series of B implies that B and its quotient rings are locally finite, and that elements of positive degree in B and in its quotient rings are not units. Applying Lemma 2.1 repeatedly, the desired Hilbert series equals that associated to the complete intersection of n (commutative) quadrics in $\mathbb{P}^{n-1}$. The result follows. ■

Some results of Levasseur in [8] that use Gelfand-Kirillov dimension (GK-dimension) will be useful in this section; the details of those results to be used later are given in Proposition 2.3. The reader is referred to [8] for definitions.

Proposition 2.3.

(a) [8, Theorem 5.3] *Suppose that B is a noetherian $\Bbbk$-algebra on which GK-dimension is exact. If B is homogeneous with respect to GK-dimension, then the set X of regular elements in B satisfies:*

$$\begin{aligned} X &= \{b \in B : \mathrm{GKdim}(B/(Bb)) < \mathrm{GKdim}(B)\} \\ &= \{b \in B : \mathrm{GKdim}(B/(bB)) < \mathrm{GKdim}(B)\}. \end{aligned}$$

(b) [8, Remark 3.4(3) and Proposition 5.9] *If B is a noetherian, Auslander-Gorenstein $\Bbbk$-algebra of finite injective dimension and finite integral GK-dimension that satisfies the Cohen-Macaulay property, then B is homogeneous with respect to GK-dimension. Additionally, if $f \in B$ is normal and regular, then $B/\langle f\rangle$ is Auslander-Gorenstein of finite injective dimension, satisfies the Cohen-Macaulay property and is homogeneous with respect to GK-dimension.*

(c) [8, Lemma 5.7 and Theorem 5.10] *Suppose B is a finitely-generated $\mathbb{N}$-graded $\Bbbk$-algebra with $\dim_{\Bbbk}(B_0) < \infty$ such that B has polynomial growth. Let $f \in B$ be a homogeneous normal element of positive degree.*

(i) *If f is not a zero divisor on a finitely generated graded B-module M, then* $\mathrm{GKdim}(M/(fM)) = \mathrm{GKdim}(M) - 1$.

(ii) *If f is regular in B, and if $B/\langle f\rangle$ is Auslander-Gorenstein of dimension ν, satisfies the Cohen-Macaulay property and has polynomial growth, then B is Auslander-Gorenstein of dimension $\nu+1$, satisfies the Cohen-Macaulay property and has polynomial growth.*

Remark 2.4. In the setting of Proposition 2.3(c)(i), where f is not a zero divisor on M, since $\mathrm{GKdim}(M/(fM)) = \mathrm{GKdim}(M) - 1$, it follows that if g is a zero divisor on M, then $\mathrm{GKdim}(M/(gM)) \geq \mathrm{GKdim}(M) - 1$.

It is well known that GK-dimension is exact on finitely generated modules over an $\mathbb{N}$-graded $\Bbbk$-algebra (c.f., [6, Theorem 6.14]).

Theorem 2.5. *Suppose B is a noetherian, $\mathbb{N}$-graded, Auslander-Gorenstein $\Bbbk$-algebra of finite injective dimension that satisfies the Cohen-Macaulay property with* $\mathrm{GKdim}(B) = n \in \mathbb{N}\setminus\{0\}$. *If $\{f_1, \ldots, f_n\}$ is a normalizing sequence in B, where each f_i is homogeneous of positive degree, then $\{f_1, \ldots, f_n\}$ is a regular sequence in B if and only if* $\dim_{\Bbbk}(B/\langle f_1, \ldots, f_n\rangle) < \infty$. *In this case, for each $k = 1, \ldots, n$, $\{f_1, \ldots, f_k\}$ is a regular sequence in B if and only if* $\mathrm{GKdim}(B/\langle f_1, \ldots, f_k\rangle) = n - k$.

Proof. For each $k = 1, \ldots, n$, write $B(k) = B/\langle f_1, \ldots, f_k\rangle$, and set $B(0) = B$.

Suppose $\{f_1, \ldots, f_n\}$ is a regular sequence. By Proposition 2.3(c), $\mathrm{GKdim}(B(k)) = n - k$ for all k, so $\dim_{\Bbbk}(B(n)) < \infty$.

To prove the converse, suppose $\{f_1, \ldots, f_k\}$ is a regular sequence for some $k = 1, \ldots, n$, but that f_{k+1} is not regular in $B(k)$. By Proposition 2.3(b), $B(k)$ is homogeneous, so by Proposition 2.3(a) and (c), $\mathrm{GKdim}(B(k+1)) = \mathrm{GKdim}(B(k)) = n - k$. It now follows, from Proposition 2.3(c) and Remark 2.4, that, for any $N = k+2, \ldots, n$, $\mathrm{GKdim}(B(N)) \geq (n-k) - (N-k-1) = n - N + 1$. Hence, $\mathrm{GKdim}(B(n)) \geq 1$, so that $\dim_{\Bbbk}(B(n)) = \infty$, which concludes the proof. ∎

Corollary 2.6. *Let $S = \Bbbk[z_1, \ldots, z_n]$ with defining relations $z_j z_i = \mu_{ij} z_i z_j$ (as in Section 1). If $\{f_1, \ldots, f_n\}$ is a normalizing sequence in S of homogeneous elements of positive degree, then $\{f_1, \ldots, f_n\}$ is a regular sequence in S if and only if* $\dim_{\Bbbk}(S/\langle f_1, \ldots, f_n\rangle) < \infty$. *In this case, for each $k = 1, \ldots, n$, $\{f_1, \ldots, f_k\}$ is a regular sequence in S if and only if* $\mathrm{GKdim}(S/\langle f_1, \ldots, f_k\rangle) = n - k$.

Proof. It is well known that S is noetherian and that $\mathrm{GKdim}(S) = n$. Since the elements $z_1, \ldots, z_n$ are normal in S and form a regular sequence, Proposition 2.3(c) implies that S is Auslander-Gorenstein of injective dimension n and satisfies the Cohen-Macaulay property. Hence, Theorem 2.5 applies to S, which completes the proof. ∎

A consequence of Corollary 3.6 below will be that the normalizing sequence $\{f_1, \ldots, f_n\}$ in Corollary 2.6 is regular if and only if the associated quadric system (in the sense of Definition 1.5) is base-point free (in the sense of Definition 1.7), thereby generalizing the notion of complete intersection to the non-commutative setting of skew polynomial rings.

3. The Geometry of Normalizing Sequences

This section explores the concept of complete intersection for normalizing sequences in a skew polynomial ring.

Let $S = \Bbbk[z_1, \ldots, z_n]$ with defining relations $z_j z_i = \mu_{ij} z_i z_j$, for $1 \leq i, j \leq n$, with the μ_{ij} as in Section 1. In particular, we continue to assume $\mu_{ii} = 1$ for all i. In the notation of Subsection 1.3, let $V^* = S_1$, so that U is the span in $V^* \otimes_{\Bbbk} V^*$ of the defining relations of S.

In the spirit of [2], we write $\Gamma_2 = \mathcal{V}(U) \subset \mathbb{P}(V) \times \mathbb{P}(V)$, and we use $\Gamma_1 \subset \mathbb{P}(V)$ to denote the image of the projection of Γ_2 onto the first copy of $\mathbb{P}(V)$. For all $i \geq 3$, let Γ_i denote all elements $(p_1, \ldots, p_i)$ of $\mathbb{P}(V) \times \cdots \times \mathbb{P}(V)$ such that $(p_j,\ p_{j+1}) \in \Gamma_2$ for all $j \in 1, \ldots, i-1$, and let

$$\Gamma = \{(p_1,\ p_2, \ldots) \in \mathbb{P}(V) \times \mathbb{P}(V) \times \cdots : (p_j,\ p_{j+1}) \in \Gamma_2 \text{ for all } j \geq 1\}.$$

Definition 3.1. [2] A right (respectively, left) *point module* over S is a cyclic graded right (respectively, left) S-module $M = \bigoplus_{i \geq 0} M_i$ such that $M = M_0 S$ (respectively, SM_0) and $\dim_{\Bbbk}(M_i) = 1$ for all i.

By [2, Proposition 3.9 and Corollary 3.13], Γ parametrizes the isomorphism classes of right (respectively, left) point modules over S as follows. Let $M = \bigoplus_{i \geq 0} M_i$ denote a right point module over S, and, for all $i \geq 0$, write $M_i = \Bbbk u_i$. For all i, j, $u_i z_j = \alpha_{ij} u_{i+1}$, where $\alpha_{ij} \in \Bbbk$. Thus, to M we may associate the point $p = (p_1, p_2, \ldots) \in \mathbb{P}(V) \times \mathbb{P}(V) \times \cdots$, where $p_{i+1} = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}) \in \mathbb{P}(V)$ for all $i \geq 0$ (so that the module action may be written $u_i z_j = z_j(p_{i+1}) u_{i+1}$ for all i, j). Moreover, p is well defined, $p \in \Gamma$, and $\Gamma_i \cong \Gamma$ for all $i \geq 1$.

Lemma 3.2. *Suppose $p = (p_1, p_2, \ldots) \in \Gamma$, and $M = M_0 S$ is the right point module over S corresponding to p. Let $f \in S_d$, where $d > 0$.*

(a) *We have $f(p_1, \ldots, p_d) = 0$ if and only if $M_0 f = 0$.*

(b) *If f is normal in S, then $f(p_1, \ldots, p_d) = 0$ if and only if $Mf = 0$.*

Proof. Clearly (b) holds if (a) holds. To prove (a), write, as above, $M = \bigoplus_{i \geq 0} \Bbbk u_i$, so that $u_i z_j = z_j(p_{i+1}) u_{i+1}$ for all i, j. Thus, $u_0 f = f(p_1, \ldots, p_d) u_d$, since $f \in S_d$, whence (a) holds. ■

Definition 3.3. If $I = \langle f_1, \ldots, f_m \rangle \subset S$, where $f_i \in S_{d_i}$, $d_i > 0$, for all $i = 1, \ldots, m$, then we say $p \in \mathcal{V}_U(I)$ if $p = (p_1, p_2, \ldots) \in \Gamma$ and if $f_i(p_1, \ldots, p_{d_i}) = 0$ for all i. In other words, by Lemma 3.2(a), $p \in \mathcal{V}_U(I)$ if and only if $M_0 f_i = 0$, for all $i = 1, \ldots, m$, where $M = M_0 S$ is the right point module over S corresponding to p.

Remark 3.4.

(a) Comparing Definition 3.3 with Definition 1.5, if $m = 1$ in Definition 3.3, and if $f_1 \in S_2$, then $\mathcal{V}_U(I)$ in Definition 3.3 is $\mathcal{V}_U(f_1)$ in Definition 1.5.

(b) Using the dual basis to $\{z_1, \ldots, z_n\}$ as a basis for V, for all $j = 1, \ldots, n$, let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{P}(V)$, where the 1 is in the jth coordinate. Owing to the defining relations of S, $e_j \in \Gamma_1$ for all $j = 1, \ldots, n$. If $f \in S_d$, where $d > 0$, and if f has no term belonging to $\Bbbk^\times z_j^d$, for some $j = 1, \ldots, n$, then $f(p) = 0$, where $p = (e_j, \ldots, e_j) \in \Gamma_d$; in this case, $p \in \mathcal{V}_U(f)$.

Let $X = \{1, z, z^2, \ldots\} \subset S$, where $z \in \{z_1, \ldots, z_n\}$. The subring, S', of SX^{-1}, consisting of the homogeneous degree-zero elements in SX^{-1}, is again a skew polynomial ring and its maximal ideals have codimension one. If $\mathfrak{m}$ is a maximal ideal in S', then $\Bbbk[z] \otimes_{\Bbbk} (S'/\mathfrak{m})$ is a right S-module and is a point module over S on which z acts faithfully. If $z = z_i$, one can realize this correspondence geometrically in affine space $\mathbb{A}^{n-1}$ and in projective space $\mathbb{P}^{n-1} = \mathbb{P}(V)$ as follows: grade S' by assigning degree one to the generators of S', then the point $(p_1, \ldots, p_{n-1}) \in \mathbb{A}^{n-1}$ given by the zero locus of $\mathfrak{m} \cap ((S')_1 + \Bbbk))$ corresponds to the point $(p_1, \ldots, p_{i-1}, 1, p_i, \ldots, p_{n-1}) \in \mathbb{P}(V)$.

Proposition 3.5. *As above, let* $S = \Bbbk[z_1, \ldots, z_n]$ *with defining relations* $z_j z_i = \mu_{ij} z_i z_j$ *for* $1 \le i, j \le n$. *Let* $I = \langle f_1, \ldots, f_n \rangle$, *where* $f_i \in S_{d_i}$, $d_i > 0$ *for all* $i = 1, \ldots, n$. *If* $\{f_1, \ldots, f_n\}$ *is a normalizing sequence in* S, *then* $\mathcal{V}_U(I)$ *is empty if and only if* $\dim_{\Bbbk}(S/I)$ *is finite.*

Proof. Let $p \in \Gamma$ and let $M(p)$ denote the right point module over S corresponding to p. If $p \in \mathcal{V}_U(I)$, then $M(p) \cong S/J$, where J is a right ideal of S and $I \subset J$. Thus, if $p \in \mathcal{V}_U(I)$, then $S/I \twoheadrightarrow M(p)$, so that $\dim_{\Bbbk}(S/I) = \infty$.

Suppose $\dim_{\Bbbk}(S/I) = \infty$. Thus, $S_d \not\subset I$ for all d, so there exists $z \in \{z_1, \ldots, z_n\}$ such that $z^t \notin I$ for all $t \in \mathbb{N}$. If no f_i has a term belonging to $\Bbbk^{\times} z^{d_i}$, then, by Remark 3.4(b), $\mathcal{V}_U(I)$ is nonempty, and the result is proved. Hence, we may assume at least one f_i has a term belonging to $\Bbbk^{\times} z^{d_i}$.

Letting $X = \{1, z, z^2, \ldots\}$, we have $X \cap I = \emptyset$. Let S' be the subring of SX^{-1} consisting of the homogeneous degree-zero elements in SX^{-1}, and let I' denote the image of I in S', and, for all $i = 1, \ldots, n$, let f_i' denote the image of f_i in S'.

If $I' = S'$, then $1 \in I'$ and, since $\{f_1, \ldots, f_n\}$ is normalizing, this would mean $z^t \in I$ for some $t \in \mathbb{N}$, which is false. Thus, $I' \neq S'$ and, viewing the generators of S' as homogeneous of degree-one, we have I' is a proper nonhomogeneous ideal in S' that is not contained in any homogeneous ideal (since some f_i' has a term that belongs to $\Bbbk^{\times}$). Since S' is noetherian, there exists a maximal ideal $\mathfrak{m}$ in S' that contains I'. As remarked earlier, $\dim(S'/\mathfrak{m}) = 1$, and $M = \Bbbk[z] \otimes_{\Bbbk} (S'/\mathfrak{m})$ is a right point module over S on which z acts faithfully. Since $I' \subset \mathfrak{m}$, f_i' vanishes on $S'/\mathfrak{m}$ for all $i = 1, \ldots, n$, and since $f_i = f_i' z^{m_i}$ for some $m_i \in \mathbb{N}$, f_i vanishes on $M_0 = \Bbbk \otimes_{\Bbbk} (S'/\mathfrak{m})$ for all $i = 1, \ldots, n$. Hence, the point in Γ that corresponds to M belongs to $\mathcal{V}_U(I)$, so that $\mathcal{V}_U(I)$ is nonempty. ■

Corollary 3.6. *As above, let* $S = \Bbbk[z_1, \ldots, z_n]$ *with defining relations* $z_j z_i = \mu_{ij} z_i z_j$ *for* $1 \le i, j \le n$. *If* $\{f_1, \ldots, f_n\}$ *is a normalizing sequence in* S *of homogeneous elements of positive degree, then the following are equivalent:*

(a) $\{f_1, \ldots, f_n\}$ *is a regular sequence in* S

(b) $\dim_{\Bbbk}(S/\langle f_1, \ldots, f_n\rangle) < \infty$

(c) *for each* $k = 1, \ldots, n$, $\mathrm{GKdim}(S/\langle f_1, \ldots, f_k\rangle) = n - k$

(d) $\mathcal{V}_U(I)$ *is empty, where* $I = \langle f_1, \ldots, f_n\rangle$.

Proof. The result follows from combining Proposition 3.5 with Corollary 2.6. ■

Definition 3.7. As above, let $S = \Bbbk[z_1, \ldots, z_n]$ with defining relations $z_j z_i = \mu_{ij} z_i z_j$ for $1 \le i, j \le n$. If $\{f_1, \ldots, f_n\}$ is a normalizing sequence in S of homogeneous elements of positive degree, then we call $S/\langle f_1, \ldots, f_n\rangle$ a *complete intersection* if any of Corollary 3.6(a)-(d) hold.

If all the $\mu_{ij} = 1$, then our definition of complete intersection reduces to the usual definition of complete intersection in the commutative setting. However, if any $\mu_{ij} \ne 1$, then possibly $\dim(\Gamma) < n - 1$, so, for each $k = 1, \ldots, n$, the dimension of $\bigcap_{i=1}^{k} \mathcal{V}_U(f_i)$, where $\{f_1, \ldots, f_n\}$ is as in Definition 3.7, need not decrease by one as k increases by one (even though the GK-dimension of the corresponding ring decreases by one each time).

4. Regular Skew Clifford Algebras

In this section, we prove our main result, Theorem 4.2, that the quadric system associated to a graded skew Clifford algebra $A = A(\mu, M_1, \ldots, M_n)$ is normalizing and base-point free (that is, a complete intersection) if and only if A is quadratic, Auslander-regular and unique up to isomorphism. In Proposition 4.5, necessary and sufficient conditions are given for a regular graded skew Clifford algebra to be a twist (see Definition 4.4) by an automorphism of a regular graded Clifford algebra. We also prove that if the quadric system associated to a regular graded skew Clifford algebra A is intersected with a "hyperplane" of $\mathcal{V}(U)$, then it yields a quadric system that corresponds to a regular graded skew Clifford subalgebra of A.

Definition 4.1. [9] If B is an Artin-Schelter regular $\mathbb{N}$-graded $\mathbb{k}$-algebra of global dimension $n \in \mathbb{N} \setminus \{0\}$ with Hilbert series $1/(1-t)^n$, then B is called a *quantum* $\mathbb{P}^{n-1}$.

By [9, Theorem 2.2], a quantum $\mathbb{P}^{n-1}$ is a quadratic algebra on n degree-one generators. By [9, Theorem 2.4], if Λ is a finite-dimensional vector space and if $B = T(\Lambda)/\langle \mathcal{W} \rangle$, where $\mathcal{W} \subset \Lambda \otimes_{\mathbb{k}} \Lambda$, is a quantum $\mathbb{P}^{n-1}$ and if $\{f_1, \ldots, f_n\} \subset \Lambda \otimes_{\mathbb{k}} \Lambda$ yields a regular normalizing sequence in B, then $T(\Lambda^*)/\langle (\mathcal{W} + \mathbb{k}f_1 + \cdots + \mathbb{k}f_n)^{\perp} \rangle$ is a quantum $\mathbb{P}^{n-1}$.

Recall that U is the span of the defining relations of the skew polynomial ring S in Subsection 1.2. The following result uses the notions of *normalizing quadric system* and *base-point free* given in Definitions 1.5, 1.11 and 1.7. We no longer assume the matrices $M_1, \ldots, M_n$ are linearly independent.

Theorem 4.2. *Let $S = T(V^*)/\langle U \rangle = \mathbb{k}[z_1, \ldots, z_n]$ with defining relations $z_j z_i = \mu_{ij} z_i z_j$, for $1 \le i, j \le n$, with the μ_{ij} as in Section 1, and, for all $k = 1, \ldots, n$, let $q_k = z^T M_k z \in S_2$ as in Section 1. A graded skew Clifford algebra $A = A(\mu, M_1, \ldots, M_n)$ is a quadratic, Auslander-regular algebra of global dimension n that satisfies the Cohen-Macaulay property with Hilbert series $1/(1-t)^n$ if and only if the quadric system associated to $\{q_1, \ldots, q_n\}$ is normalizing and base-point free; in this case, A is a quantum $\mathbb{P}^{n-1}$ and is a noetherian domain and is unique up to isomorphism.*

Proof. Suppose A is a quadratic, Auslander-regular algebra of global dimension n that satisfies the Cohen-Macaulay property with Hilbert series $1/(1-t)^n$. By Lemma 1.13, $M_1, \ldots, M_n$ are linearly independent. Moreover, $A = T(V)/\langle W \rangle$. There are elements $r_k^\star \in S_2$ that are dual to the r_k (as in Remark 1.17), and $\sum_{k=1}^n \mathbb{k} r_k^\star = \sum_{k=1}^n \mathbb{k} q_k$, by Corollary 1.16.

Moreover, $A/\langle r_1, \ldots, r_n\rangle$ is the Koszul dual of S, and, since S is Koszul, the Hilbert series of $A/\langle r_1, \ldots, r_n\rangle$ is $(1+t)^n$. Since $\{r_1, \ldots, r_n\}$ is a normalizing sequence in A, comparison of Hilbert series implies that it is also a regular sequence in A. By [8, Proposition 3.5(a) and Theorem 6.3], A is noetherian and Artin-Schelter regular, so by [9, Theorem 2.2], A is Koszul. Moreover, by [9, Lemma 1.3], $\{r_n^\star, \ldots, r_1^\star\}$ is a normalizing sequence in S. Hence, the quadric system associated to $q_1, \ldots, q_n$ is normalizing, and the Koszul dual of A is $T(V^*)/\langle W^\perp\rangle = S/\langle q_1, \ldots, q_n\rangle$, which has finite dimension since A is Koszul. Thus, Corollary 3.6 implies that $\mathcal{V}_{_U}(I)$ is empty where $I = \langle q_1, \ldots, q_n\rangle$. Since $q_i \in S_2$ for all i, $|\mathcal{V}_{_U}(I)| = |\bigcap_{i=1}^n \mathcal{V}_{_U}(q_i)|$, so the associated quadric system is base-point free.

Conversely, suppose the quadric system is normalizing and base-point free. In particular, by Corollary 3.6, $q_1, \ldots, q_n$ are linearly independent, and so, by Lemma 1.3, $M_1, \ldots, M_n$ are linearly independent. By Corollary 3.6, there exists a regular normalizing sequence $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\} \subset S_2$ such that $\sum_{k=1}^n \Bbbk\mathfrak{q}_k = \sum_{k=1}^n \Bbbk q_k$. By [9, Corollary 1.4], there exists a regular normalizing sequence $\{\mathfrak{q}_n^\star, \ldots, \mathfrak{q}_1^\star\}$ of homogeneous degree-two elements in $T(V)/\langle W\rangle$. Since $\sum_{k=1}^n \Bbbk q_k = \sum_{k=1}^n \Bbbk r_k^\star$, we have $\sum_{k=1}^n \Bbbk\mathfrak{q}_k^\star = \sum_{k=1}^n \Bbbk r_k = V''$. It follows that the relations in Definition 1.12(b) are a consequence of the relations in Definition 1.12(a), and so $A = T(V)/\langle W\rangle$ and is quadratic. Moreover, by Corollary 1.16, $T(V)/\langle W\rangle$ is the Koszul dual of $S/\langle \mathfrak{q}_1, \ldots, \mathfrak{q}_n\rangle$, so, by [9, Theorem 2.4], $T(V)/\langle W\rangle$ is a quantum $\mathbb{P}^{n-1}$. Since $A = T(V)/\langle W\rangle$ and since $A/\langle \mathfrak{q}_n^\star, \ldots, \mathfrak{q}_1^\star\rangle$ is Auslander-Gorenstein and satisfies the Cohen-Macaulay property, so does A (by Proposition 2.3(c)), and so A is Auslander-regular.

By [8, Proposition 3.5(a) and Theorem 4.8], A is a noetherian domain. Since $A = T(V)/\langle W\rangle$, A is unique up to isomorphism. ■

Corollary 4.3. *If B is a graded Artin-Schelter regular algebra and if gldim$(B) \le 2$, then B is a graded skew Clifford algebra.*

Proof. If gldim$(B) = 1$, then $B \cong \Bbbk[x]$, so we take $n = 1$, $x_1 = x$, $\mu = 1$ and $M_1 = 1$. If gldim$(B) = 2$, then $B \cong \Bbbk\langle x_1, x_2\rangle/\langle f\rangle$, where either (i) $f = x_1x_2 + \lambda x_2x_1$, $\lambda \in \Bbbk^\times$, or

(ii) $f = x_1x_2 - x_2x_1 - x_2^2$ ([1]). For both cases, we may take $n = 2$, $M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, and $M_2 = \begin{bmatrix} 0 & \nu \\ \mu_{21}\nu & 2 \end{bmatrix}$, where in (i), $(\mu_{12}, \nu) = (\lambda, 0)$, and, in (ii), $(\mu_{12}, \nu) = (-1, 1)$. Thus, $q_1 = 2z_1^2$ and $q_2 = 2(z_2^2 + \nu z_1 z_2)$, and, in $S/\langle q_1 \rangle$, $z_2 q_2 = q_2(\mu_{21}(1-\mu_{21})\nu z_1 + z_2)$ and $q_2 z_2 = ((1-\mu_{12})\nu z_1 + z_2)q_2$, so $\{q_1, q_2\}$ yields a normalizing quadric system. Since $\{q_1, q_2\}$ is also base-point free, the result follows from Theorem 4.2. ■

If A is a graded skew Clifford algebra and if S is a skew polynomial ring related to A as in Theorem 4.2, one should note that S need not be unique, even if A is quadratic; this is illustrated below while using the notion of *twist*, which is defined as follows.

Definition 4.4. [3, §8] Let $B = \bigoplus_{n \geq 0} B_n$ be a quadratic algebra and let ϕ be a graded degree-zero automorphism of B. The twist B^ϕ of B by ϕ is the vector space $\bigoplus_{n \geq 0} B_n$ with a new multiplication $*$ defined as follows: if $a, b \in B_1$, then $a * b = a\phi(b)$, where the right-hand side is computed using the original multiplication in B.

If $n = 3$ and if we take $\mu_{12} = \mu_{13} = 1 \neq \mu_{23}$, then S is not a twist of a polynomial ring, and $\mathfrak{Q} = \{z_2z_3, z_1^2, z_2^2 + z_3^2\} \subset S$ determines a regular graded skew Clifford algebra A as in Theorem 4.2. However, $S/\langle \mathfrak{Q} \rangle$ is commutative, and so A is a graded Clifford algebra; in particular, S may be replaced with a polynomial ring.

In general, if S and A are as in Theorem 4.2, and if $\mathfrak{Q} \subset S_2$ is a regular normalizing sequence of length n such that $S/\langle \mathfrak{Q} \rangle$ twists to a commutative algebra (that is, A is a twist of a graded Clifford algebra), then, by [10, 15], it is conceivable that there is no choice for S that twists to a polynomial ring. We address this issue in our next result. It should be noted that we only consider automorphisms of graded algebras that are graded of degree zero. For results concerning twists of graded algebras, the reader is referred to [3, §8] and [18]. In particular, twisting by an automorphism is symmetric and reflexive, but is not transitive in general, and twisting a quadratic algebra B to a quadratic algebra B' induces a twist of the Koszul dual of B' to the Koszul dual of B.

Proposition 4.5. *Let $n \in \mathbb{N} \setminus \{0\}$ and let S denote a skew polynomial ring on n generators. Suppose there is a regular normalizing sequence $\mathfrak{Q}$ in S corresponding to a regular graded skew Clifford algebra A of global dimension n, as in Theorem 4.2. If S is a twist of a polynomial ring by an automorphism, then A is a twist by an automorphism of a regular graded Clifford algebra. Conversely, if A is a twist by an automorphism of a regular graded Clifford algebra, then there exists a choice for S so that S is a twist of a polynomial ring by an automorphism.*

Proof. Suppose S is a twist of a polynomial ring, R, by an automorphism. Since $\mathfrak{Q}$ and the defining relations of S are given by homogeneous degree-2 elements and, since any automorphism of R acts linearly on R_1, $\mathfrak{Q}$ determines elements in R_2 that yield an n-dimensional quadric system in $\mathbb{P}^{n-1}$ that is base-point free. Hence, under the given hypotheses, the Koszul dual of A twists to the Koszul dual of a regular graded Clifford algebra, and so A is a twist of a regular graded Clifford algebra.

Conversely, suppose A is a twist by an automorphism of a regular graded Clifford algebra C. It follows that the Koszul dual, D, of C is a twist by an automorphism, τ, of the Koszul dual, B, of A. The algebra D is commutative, $S \twoheadrightarrow B$, and, for all i, $D_i = B_i$. For all $b \in B_1$, we write b^τ for $\tau(b)$. We denote the multiplication in D by $*$ and write the twist as $f * g = fg^\tau$ for all $f, g \in D_1$. The commutativity of D implies that, in B,

$$fg^\tau = gf^\tau \qquad \text{and} \qquad fg = g^{\tau^{-1}} f^\tau, \tag{\dagger}$$

for all $f, g \in B_1$. The result holds if $n < 3$, so we may assume $n \geq 3$.

Claim 1: If $z \in B_1$ is normal in B, then z^τ is normal in B.

Proof: Since τ is an automorphism of B, application of τ to $Bz = zB$ gives $Bz^\tau = z^\tau B$.

Claim 2: Eigenvectors of τ in B_1 are normal elements of B.

Proof: The claim follows from (†).

Suppose there exist linearly independent, normal elements $z_1, \ldots, z_n \in B_1$ such that $z_i^\tau = \lambda_i z_i$, where $\lambda_i \in \Bbbk^\times$ for all i. By (†), we have $z_j z_i = \lambda_i^{-1} \lambda_j z_i z_j$ in B for all i, j. It follows that we may rechoose S to be the algebra generated by $z_1, \dots, z_n$ with defining relations $z_j z_i = \lambda_i^{-1}\lambda_j z_i z_j$ for all i, j, and this choice of S twists, by τ, to a polynomial ring.

Returning to the general setting, since τ acts linearly on B_1, there is an eigenvector $z_1 \in B_1^\times$. By scaling τ if needed, we may assume $z_1^\tau = z_1$. By Claim 2, z_1 is normal in B. Since B_1 contains n linearly independent normal elements, we may extend $\{z_1\}$ to a basis $\{z_1, \dots, z_n\}$ for B_1 consisting of normal elements, and we may write $z_j z_i = \mu_{ij} z_i z_j$ for all i, j where $\mu_{ij} \in \Bbbk^\times$, for all i, j, as in Subsection 1.2. Moreover, by the preceding paragraph, we may assume that $z_n^\tau \notin \Bbbk z_n$ (after reordering the z_i, if needed).

Claim 3: There exists $w \in B_1^\times$ such that $z_j^\tau \in \Bbbk^\times z_j + \Bbbk w$ for all $j \geq 2$, $w^\tau \in \Bbbk^\times w$ and $w z_1 = 0 = z_1 w$.

Proof: By (†), we have

$$z_i^{\tau^{-1}} z_j^\tau = z_j z_i = \mu_{ij} z_i z_j = \mu_{ij} z_j^{\tau^{-1}} z_i^\tau, \qquad (*)$$

for all i, j. Since $z_1^\tau = z_1$, setting $i = 1$ in $(*)$ implies that

$$(z_j - \mu_{1j} z_j^{\tau^{-1}}) z_1 = 0 = z_1 (z_j^\tau - \mu_{1j} z_j),$$

for all $j \geq 1$. However, since A is a twist by an automorphism of a regular graded Clifford algebra, the point scheme of A is the graph of an automorphism, and so there exists at most one linearly independent element $w \in B_1^\times$ such that $z_1 w = 0$ in B. Since $\tau \in \mathrm{Aut}(B)$, w satisfies $z_1 w^\tau = z_1^\tau w^\tau = \tau(z_1 w) = 0$. Claim 3 follows.

By Claim 2, the element w, from Claim 3, is normal in B.

If w and z_1 are linearly independent, we may rechoose $z_2, \dots, z_n$ and the μ_{ij} so that $z_2 = w$ and $z_n^\tau \notin \Bbbk z_n$, and write $z_2^\tau = \lambda_2 z_2$ where $\lambda_2 \in \Bbbk^\times$. Using $(*)$ with $i = 2$ yields

$$\lambda_2^{-1} z_2 z_j^\tau = z_j z_2 = \mu_{2j} z_2 z_j = \lambda_2 \mu_{2j} z_j^{\tau^{-1}} z_2,$$

for all $j \geq 3$, from which we obtain

$$z_2(\lambda_2^{-1} z_j^\tau - \mu_{2j} z_j) = 0 = (z_j - \lambda_2 \mu_{2j} z_j^{\tau^{-1}}) z_2,$$

for all $j \geq 3$. As above, there is at most one linearly independent element $w' \in B_1$ such that $z_2 w' = 0$. It follows that $w' \in \Bbbk z_1$ and that $z_j^\tau \in \Bbbk^\times z_j + \Bbbk z_1$ for all $j \geq 3$. However, z_1 and z_2 are linearly independent and $z_j^\tau \in \Bbbk^\times z_j + \Bbbk z_2$ for all $j \geq 3$, so $z_j^\tau \in \Bbbk^\times z_j$ for all j, which contradicts our assumption that $z_n^\tau \notin \Bbbk z_n$. Hence, w and z_1 are linearly dependent, and $z_j^\tau \in \Bbbk^\times z_j + \Bbbk z_1$ for all $j \geq 2$ and $z_1^2 = 0$.

We will now prove that $\Bbbk z_1$ is the only eigenspace of τ in B_1. If this is false, then there exists $z \in B_1 \setminus \Bbbk z_1$ such that $z^\tau \in \Bbbk z$ (so z is normal, by Claim 1). This allows us to rechoose $z_2, \ldots, z_n$ (and the μ_{ij}) so that z_2 is z and $z_n^\tau \notin \Bbbk z_n$, and apply $(*)$ with $i = 2$ to obtain either (i) $z_j^\tau \in \Bbbk^\times z_j + \Bbbk z_3$ for all $j \geq 4$ (after rechoosing z_3, if needed) and $z_j^\tau \in \Bbbk^\times z_j$ for $j = 1, 2, 3$ and $z_2 z_3 = 0 = z_3 z_2$, or (ii) $z_j^\tau \in \Bbbk^\times z_j + \Bbbk z_2$ for all $j \geq 3$ and $z_j^\tau \in \Bbbk^\times z_j$ for $j = 1, 2$ and $z_2^2 = 0$. If (i) (respectively, (ii)) occurs, then, since z_1 and z_3 (respectively, z_2) are linearly independent and since $z_j^\tau \in \Bbbk^\times z_j + \Bbbk z_1$ for all $j \geq 2$, it follows that $z_j^\tau \in \Bbbk^\times z_j$ for all j, which gives a contradiction as before.

It follows that $\Bbbk z_1$ is the only eigenspace of τ in B_1, and that we may assume $z_j^\tau \in \Bbbk^\times z_j + \Bbbk^\times z_1$ for all $j \geq 2$ and that $z_1^2 = 0$. For each $j \geq 2$, we write $z_j^\tau = \alpha_j z_j + \beta_j z_1$, where $\alpha_j, \beta_j \in \Bbbk^\times$. By Claim 1, z_j^τ is normal in B for all j, so

$$z_k(\alpha_j z_j + \beta_j z_1) = z_k z_j^\tau \in \Bbbk z_j^\tau z_k = \Bbbk(\alpha_j z_j + \beta_j z_1) z_k,$$

for all $k, j \geq 2$. However,

$$z_k(\alpha_j z_j + \beta_j z_1) = (\alpha_j \mu_{jk} z_j + \beta_j \mu_{1k} z_1) z_k,$$

for all $k, j \geq 2$. Thus, setting $j = k$, we have that $\mu_{1k} = 1$ for all k (since $z_1^2 = 0$ implies that $z_k z_1 \neq 0 \neq z_1 z_k$ for all $k \neq 1$). Hence, for all $j, k \geq 2$, z_j and z_k commute (though, possibly, $z_k z_j = 0 = z_j z_k$ for some k, j). It follows that B is commutative, so that S may be chosen to be a polynomial ring. ■

The following result relates a regular graded skew Clifford subalgebra of a regular graded skew Clifford algebra to the intersection of a quadric system with a codimension-one linear subspace (that is, with a "hyperplane"). The result and its proof are motivated by [14, Proposition 1.9].

Theorem 4.6. *Let $n \in \mathbb{N} \setminus \{0\}$ and let S denote a skew polynomial ring on n generators. Suppose there is a regular normalizing sequence $\mathfrak{Q}$ in S corresponding to a regular graded skew Clifford algebra A of global dimension n, as in Theorem 4.2. Let $\Bbbk\mathfrak{Q}$ denote the span of the elements of $\mathfrak{Q}$ and let $\eta \in S_1^\times$ denote a normal element. If the image of $\Bbbk\mathfrak{Q}$ in $S/\langle\eta\rangle$ has dimension at most $n-1$, then it contains a regular normalizing sequence of $S/\langle\eta\rangle$ that corresponds to a regular graded skew Clifford subalgebra of A of global dimension $n-1$.*

Proof. The image of $\mathfrak{Q}$ in $S/\langle\eta\rangle$ is a normalizing sequence, and, since η is normal and belongs to S_1, $S/\langle\eta\rangle$ is a skew polynomial ring on $n-1$ generators. By Corollary 3.6, $\mathcal{V}_U(\mathfrak{Q})$ is empty, so the zero locus of the image of $\mathfrak{Q}$ in $S/\langle\eta\rangle$ is also empty. Thus, if the image of $\Bbbk\mathfrak{Q}$ in $S/\langle\eta\rangle$ has dimension at most $n-1$, then, by Corollary 3.6, the image of $\Bbbk\mathfrak{Q}$ in $S/\langle\eta\rangle$ has dimension equal to $n-1$, and it contains a regular normalizing sequence of $S/\langle\eta\rangle$ that corresponds to a regular graded skew Clifford algebra A' of global dimension $n-1$. It follows that the (conceivably nonquadratic) subalgebra, B, of A generated by $\eta^\perp$ is a homomorphic image of A', so $H_B(t) \leq 1/(1-t)^{n-1}$.

Writing $\mathfrak{Q} = \{q_1, \ldots, q_n\}$, the dimension hypothesis implies that there exists $m \in \{1, \ldots, n\}$ and $z \in S_1^\times$ such that $q_m - \eta z \in \Bbbk q_1 + \cdots + \Bbbk q_{m-1}$. Thus, in $S/\langle q_1, \ldots, q_{m-1}\rangle$, we have $q_m = \eta z$, so that we may assume $q_m = \eta z$ in S for some $z \in S_1^\times$. Moreover, $\eta = \sum_{k=1}^n \alpha_k z_k$, where $\alpha_k \in \Bbbk$ for all k. If $\eta \notin \Bbbk^\times z_\ell$ for all ℓ, then, since η is normal, $\mu_{ij} = 1$ for all i, j such that $\alpha_i \alpha_j \neq 0$. It follows that we may change basis for S_1 and assume $\eta = z_m$, so that $q_m = z_m z$. In order to simplify notation, for the rest of the proof we will assume $m = n$, but the argument holds for any $m \in \{1, \ldots, n\}$. Thus, we assume $q_n \in S_1^\times z_n$.

In A, for $1 \leq i, j \leq n$, we have $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^n (M_k)_{ij} y_k$, where M_k is the μ-symmetric $n \times n$ matrix corresponding to q_k, for $k = 1, \ldots, n$. Since $q_n \in S_1^\times z_n$, $(M_n)_{ij} = 0$

for $1 \leq i, j \leq n-1$, so that for $1 \leq i, j \leq n-1$, we have

$$x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^{n-1} (M_k)_{ij} y_k. \qquad (*)$$

Moreover, for $1 \leq k \leq n-1$, we may replace each M_k with the $(n-1) \times (n-1)$ submatrix formed from the first $n-1$ rows and first $n-1$ columns of M_k and retain the equations $(*)$. Since $z_n^{\perp} = \Bbbk x_1 + \cdots + \Bbbk x_{n-1}$, it follows that $(*)$ gives the degree-two relations of B and of A'. Hence, $y_1, \ldots, y_{n-1} \in B_2 = (A')_2$.

Suppose $(M_n)_{nn} \neq 0$. Thus, $y_n \in \Bbbk^\times x_n^2 + B_2$, and $x_n x_i \in \Bbbk x_n^2 + B_1 x_n + B_2$, for $1 \leq i \leq n-1$. Hence, A has a spanning set $\{\mathfrak{X}\, x_n^j\}_{j=0}^{\infty}$, where $\mathfrak{X}$ is a spanning set for B, and so $H_B(t) \geq 1/(1-t)^{n-1}$, giving $B = A'$.

Suppose $(M_n)_{nn} = 0$. Thus, $x_n^2 \in B_2$, and, since $q_n \in S_1^\times z_n$, there exists $k \in \{1, \ldots, n-1\}$ such that $x_k x_n + \mu_{kn} x_n x_k \in B_2 + \Bbbk^\times y_n$. Fix such a k and let $X_n = x_k + x_n$. Since $x_n^2 \in B_2$, a computation in A verifies that $\Bbbk y_n + \sum_{i=1}^{n-1} \Bbbk X_n x_i \subset \Bbbk X_n^2 + B_1 X_n + B_2$. It follows that A has a spanning set $\{\mathfrak{X}\, X_n^j\}_{j=0}^{\infty}$, where $\mathfrak{X}$ is a spanning set for B, and so $H_B(t) \geq 1/(1-t)^{n-1}$, giving $B = A'$ as before. ■

In the proof of Theorem 4.6, if all the $\mu_{ij} = 1$, then it is possible to change basis for S_1 to obtain $(M_n)_{nn} \neq 0$; in this case, the proof of Theorem 4.6 is similar to that of [14, Proposition 1.9].

5. Examples of Regular Skew Clifford Algebras

In this section, we construct some examples of regular graded skew Clifford algebras A of global dimension four. Using the results of this article, the method in each case entails starting with a skew polynomial ring S on four generators and then finding a normalizing quadric system $\mathfrak{Q}$ of four quadrics such that $\mathfrak{Q}$ is base-point free. The regular graded skew Clifford algebra A will be the Koszul dual of $S/\langle q_1, \ldots, q_4 \rangle$, where $\{q_1, \ldots, q_4\}$ gives $\mathfrak{Q}$. Owing to Proposition 4.5, we seek such an S that is not a twist of a polynomial ring.

To some extent, our methods allow one to control the number of point modules over A while constructing A; the idea behind this is described in the following two results.

Lemma 5.1. *In the notation of Subsection 1.3, let A denote a graded skew Clifford algebra and let $\mathcal{P}$ denote the set of pure tensors in $\mathbb{P}(W^{\perp})$; that is, $\mathcal{P} = \{a \otimes b \in \mathbb{P}(W^{\perp}) : a, b \in T(V^*)_1\}$. If A is regular, then the set of isomorphism classes of truncated point modules over A of length three is in one-to-one correspondence with $\mathcal{P}$, and the zero locus in $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ of the defining relations of A is given by $\{(a,\ b) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) : a \otimes b \in \mathcal{P}\}$.*

Proof. By [2, Section 3], the set of isomorphism classes of truncated point modules over A of length three is in one-to-one correspondence with the zero locus in $\mathbb{P}(V^*)\times\mathbb{P}(V^*)$ of the defining relations of A. The latter is $\mathcal{V}(W) = \{(a,\ b) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) : f(a, b) = 0 \ \text{ for all } f \in W\}$, which is clearly in one-to-one correspondence with $\mathcal{P}$. ■

Corollary 5.2. *In the setting of Lemma 5.1, if A is regular of global dimension at most four, then $\mathcal{P}$ parametrizes the set of isomorphism classes of point modules over A.*

Proof. By [2], if $\text{gldim}(A) \leq 3$, then the truncated point modules over A of length three can be extended to point modules, and the same holds by [10, Theorem 1.4] if $\text{gldim}(A) = 4$. ■

We conjecture that the restriction on the value of the global dimension in the preceding result is unnecessary.

In the following examples, we let $n = 4$ in each case, and let $z_1, \ldots, z_4$ generate S as in Subsection 1.2. Given that a regular graded skew Clifford algebra A is a domain, by Theorem 4.2, the dimension of its line scheme may be found by computing the dimension of the scheme of rank-two tensors in the span of the defining relations of A; for methods to compute the line scheme itself the reader is referred to [11, 12]. Since computer-algebra programs (e.g., Mathematica or Maple) were used in Examples 1-3 to find the minors of certain matrices and to generate Gröbner bases for certain systems of polynomial equations, we assume $\text{char}(\Bbbk) = 0$ in those examples.

Example 1

Take $q_1 = z_1z_2$, $q_2 = z_3z_4$, $q_3 \in \Bbbk^\times z_1^2 + \Bbbk^\times z_3^2 + \Bbbk^\times z_2z_4$, $q_4 \in \Bbbk^\times z_2^2 + \Bbbk^\times z_4^2 + \Bbbk^\times z_1z_3$. From requiring that $\{q_1, \ldots, q_4\}$ is a normalizing sequence, we obtain equations for the μ_{ij} as follows: $\pm 1 = \mu_{13} = \mu_{24} = (\mu_{14})^2 = (\mu_{23})^2$. Any base-point p of the quadric system would have the form $p = ((\alpha_1, \ldots, \alpha_4), (\beta_1, \ldots, \beta_4)) \in \mathbb{P}^3 \times \mathbb{P}^3$. Normality of the z_i in S implies that, for any i, $\alpha_i = 0$ if and only if $\beta_i = 0$. Applying q_1 and q_2 to p implies $\alpha_1\beta_2 = 0 = \alpha_3\beta_4$, and applying q_3 and q_4 to p yields no solution. Hence, with the equations for the μ_{ij} satisfied, the resulting quadric system is normalizing and base-point free, and so the corresponding graded skew Clifford algebra is quadratic and regular of global dimension four. In particular, the solution $-1 = \mu_{13} = \mu_{24}$, $\mu_{14} = \mu_{23} = i$, where $i^2 = -1$, together with $q_3 = z_1^2 + z_3^2 + z_2z_4$ and $q_4 = z_2^2 + z_4^2 + z_1z_3$, yields a regular graded skew Clifford algebra that is an algebra presented in [9] and which has exactly twenty nonisomorphic point modules and a line scheme of dimension one. The algebra in [9] was found without using geometry; in fact, in that work, W. Schelter's computer program *Affine* was used to verify the Hilbert series of $S/\langle q_1, \ldots, q_4\rangle$. However, computer algorithms can become limited when one wishes to incorporate parameters into the computation, whereas the discussion above shows that any nonzero coefficients used in the expressions for q_3 and q_4 will produce a regular graded skew Clifford algebra. We will now embed the algebra of [9] into a family of regular algebras exhibiting similar geometric properties as that in [9].

Motivated by the algebra in [9], we assign $-1 = \mu_{13} = \mu_{24}$, $\mu_{14} = i$ and $\mu_{23} = j$, where $i^2 = -1$ and $j = \pm i$, and take $q_3 = z_1^2 + z_3^2 + \gamma z_2z_4$ and $q_4 = z_2^2 + z_4^2 + z_1z_3$, where $\gamma \in \Bbbk^\times$ (nonzero coefficients besides γ in q_3 and q_4 may be taken to be one by a diagonal change of basis of S_1). The corresponding regular graded skew Clifford algebra A is the $\Bbbk$-algebra on $x_1, \ldots, x_4$ with defining relations:

$$\begin{array}{lll} x_4x_1 = ix_1x_4, & x_3^2 = x_1^2, & x_3x_1 = x_1x_3 - x_2^2, \\ x_3x_2 = jx_2x_3, & x_4^2 = x_2^2, & x_4x_2 = x_2x_4 - \gamma x_1^2, \end{array} \qquad (*)$$

where $\gamma \in \Bbbk^\times$. However, if $j = -i$, then S is a twist of the polynomial ring, and so A, in this case, is a twist of a regular graded Clifford algebra (by Proposition 4.5). If $j = i$, we denote the family of regular algebras given by the relations $(*)$ as $A(\gamma)$, and note that $A(1)$ is the algebra in [9], and that $A(\gamma) \cong A(-\gamma)$ for all $\gamma \in \Bbbk^\times$. Computer calculations verify that, for all $\gamma \in \Bbbk^\times$, $A(\gamma)$ has a line scheme of dimension one and a finite point scheme; in fact, if $\gamma^2 \neq 4$, then $A(\gamma)$ has exactly twenty nonisomorphic point modules, whereas $A(\pm 2)$ has exactly twelve nonisomorphic point modules. Hence, $A(\gamma)$ is not isomorphic to, nor a twist of, a graded Clifford algebra (since the latter has a line scheme of dimension at least two). It follows that the algebras $A(\gamma)$ are all candidates for generic regular algebras of global dimension four.

The other solutions for the μ_{ij} not considered above yield regular graded skew Clifford algebras that are isomorphic to twists (by an automorphism) of the algebras discussed above.

Example 2

Consider the quadric system $\mathfrak{Q}(\gamma)$ that corresponds to the algebras $A(\gamma)$ in Example 1, and map $z_1 \mapsto 0$. By Theorem 4.6, this yields a quadric system $\mathfrak{Q}'(\gamma)$ that corresponds to a graded skew Clifford algebra of global dimension three that is a subalgebra of $A(\gamma)$. We seek a quadric system $\tilde{\mathfrak{Q}}$ that corresponds to a graded skew Clifford algebra of global dimension four and which yields $\mathfrak{Q}'(\gamma)$ on mapping $z_1 \mapsto 0$. Hence, we take $q_1 = z_1 z_2$, $q_2 = z_3 z_4$, $q_3 = z_3^2 + \gamma z_2 z_4 + \alpha_1 z_1^2 + \alpha_2 z_1 z_4$ and $q_4 = z_2^2 + z_4^2 + \beta_1 z_1^2 + \beta_2 z_1 z_3$, where γ, α_i, $\beta_i \in \Bbbk$ for $i = 1, 2$. Since the q_i have changed from those in Example 1, we seek values for the μ_{ij} that allows $\{q_1, \dots, q_4\}$ to be a normalizing sequence, and there is more than one choice that suffices. We take $\mu_{13} = \mu_{14} = \mu_{24} = -\mu_{23} = 1$. With this choice, the quadric system is base-point free if and only if $\gamma(\alpha_1^2 + \alpha_2^2\beta_1)(\beta_1^2 + \beta_2^2\alpha_1) \neq 0$.

In the special case that $(\alpha_2, \beta_1) = (0, 0)$, the resulting graded skew Clifford algebra is isomorphic to a twist (by an automorphism) of an algebra $A(\gamma')$, for some $\gamma' \in \Bbbk^\times$, from Example 1.

Henceforth, we assume $\gamma = 1$ and $\alpha_2(\alpha_2 - 1) = 0$. The corresponding regular graded skew Clifford algebra $A(\alpha_1, \alpha_2, \beta_1, \beta_2)$ is the $\Bbbk$-algebra on $x_1, \ldots, x_4$ with defining relations:

$$\begin{aligned} x_3x_1 + x_1x_3 &= \beta_2 x_2^2, & x_2x_3 &= x_3x_2, \\ x_4x_1 + x_1x_4 &= \alpha_2 x_3^2, & \alpha_1 x_3^2 + \beta_1 x_2^2 &= x_1^2, \\ x_4x_2 + x_2x_4 &= x_3^2, & x_2^2 &= x_4^2, \end{aligned}$$

where $\alpha_2(\alpha_2 - 1) = 0$ and $(\alpha_1^2 + \alpha_2^2\beta_1)(\beta_1^2 + \beta_2^2\alpha_1) \neq 0$. Under these conditions, a computer calculation confirms that $A(\alpha_1, \alpha_2, \beta_1, \beta_2)$ has a line scheme of dimension one, and so $A(\alpha_1, \alpha_2, \beta_1, \beta_2)$ is not isomorphic to, nor a twist of, a graded Clifford algebra. Moreover, another computer calculation verifies that generic values of $\alpha_1, \beta_1, \beta_2$ yield an algebra that has exactly twenty nonisomorphic point modules, while certain values of the parameters yield an algebra with fewer nonisomorphic point modules.

Table 1 lists some values of the parameters where we have computed the total number, N, of nonisomorphic point modules over $A(\alpha_1, \alpha_2, \beta_1, \beta_2)$. Owing to the values of N in rows iii-viii of

	$A(\alpha_1, \alpha_2, \beta_1, \beta_2)$	N
i	$A(\alpha_1, 0, 2, 0)$ where $\alpha_1(1 - \alpha_1^2) \neq 0$	20
ii	$A(\alpha_1, 1, \beta_1, 0)$ where α_1, β_1 generic with $\alpha_1\beta_1(\alpha_1^2 + \beta_1) \neq 0$	20
iii	$A(\alpha_1, 1, 1 - 2\alpha_1, 2\sqrt{2})$ where α_1 generic with $\alpha_1(16\alpha_1 + 9)(\alpha_1 - 1)(2\alpha_1 + 1) \neq 0$	17
iv	$A(0, 1, \beta_1, 0)$ where $\beta_1 \neq 0, -1$	16
v	$A(-\frac{9}{16}, 1, \frac{17}{8}, 2\sqrt{2})$	15
vi	$A(1, 0, 2, 0)$	14
vii	$A(0, 1, 1, 2\sqrt{2})$	13
viii	$A(0, 1, -1, 0)$	8

TABLE 1. Some Regular Algebras from Example 2

Table 1, the algebras in those rows are clearly not isomorphic to, nor twists of, the algebras in Example 1. Focusing on rows i and vi of Table 1, we find that $A(\alpha_1, 0, 2, 0) \cong A(-\alpha_1, 0, 2, 0)$, for all $\alpha_1 \in \Bbbk^\times$, and that, at $\alpha_1 = \pm 1$, we obtain the algebra in row vi. The parameter values in Table 1 are not exhaustive, so other values of N are conceivably possible for the algebras $A(\alpha_1, \alpha_2, \beta_1, \beta_2)$, as well as other parameter values possibly yielding $N = 20$.

Since the regular algebras in Table 1 have finite point schemes and 1-dimensional line schemes, those in rows i and ii are candidates for generic regular algebras of global dimension four.

Example 3

Let $q_1 = z_1 z_2$, $q_2 = z_3^2$, $q_3 = z_1^2 - z_2 z_4$ and $q_4 = z_2^2 + z_4^2 - z_2 z_3$. The normalizing property implies $(\mu_{34})^2 = \mu_{23} = 1$ and $\mu_{34} = \mu_{24} = (\mu_{14})^2 = (\mu_{13})^2$, in which case the quadric system is normalizing and base-point free. Hence, the corresponding graded skew Clifford algebra A is quadratic and regular of global dimension four, and is the $\Bbbk$-algebra on $x_1, \ldots, x_4$ with defining relations:

$$\begin{array}{lll} x_1 x_3 = -\mu_{13} x_3 x_1, & x_3 x_4 = -\mu_{34} x_4 x_3, & x_2 x_3 + x_3 x_2 = -x_4^2, \\ x_1 x_4 = -\mu_{14} x_4 x_1, & x_4^2 = x_2^2, & x_2 x_4 + \mu_{24} x_4 x_2 = -x_1^2, \end{array}$$

where the μ_{ij} satisfy the above equations. The line scheme of A has dimension at least two. The algebra A has finitely many point modules if and only if $\mu_{13}\mu_{34} + \mu_{14} \neq 0$, and, in this case, A has exactly five nonisomorphic point modules. However, since $\mu_{34} = \pm 1$, this latter case requires $\mu_{13} = \pm \mu_{14}$, which forces the skew polynomial ring S to be a twist of the polynomial ring, so that, in this case, A is a twist of a regular graded Clifford algebra. On the other hand, if $\mu_{13}\mu_{34} + \mu_{14} = 0$, then A has exactly a one-parameter family of nonisomorphic point modules. In this latter case, the point variety is the intersection of three cubic surfaces in $\mathbb{P}^3$ and is not a (commutative) complete intersection.

Claim: If $\mu_{13}\mu_{34} + \mu_{14} = 0$, then A is not a twist by an automorphism of a graded Clifford algebra.

Proof: Suppose A is a twist by an automorphism of a graded Clifford algebra C. It follows that the Koszul dual, D, of C is commutative and is a twist by an automorphism, τ, of the Koszul dual, B, of A. Following the proof of Proposition 4.5, we write $\tau(b) = b^\tau$ for all $b \in B_1$ and denote the multiplication in D by $*$, and we write the twist as $f * g = fg^\tau$, for all $f, g \in D_1 = B_1$. In particular, (†) from the proof of Proposition 4.5 holds. Thus, writing $z_1^\tau = \sum_i \alpha_i z_i$, where $\alpha_i \in \Bbbk$ for all i, we have that, in B,

$$z_1 z_3^\tau = z_3 z_1^\tau = \sum_i \alpha_i z_3 z_i.$$

Given the relations in B and that z_1 is normal in B, it follows that $\alpha_2 = 0 = \alpha_4$. Similarly,

$$z_1 z_2^\tau = z_2 z_1^\tau = z_2(\alpha_1 z_1 + \alpha_3 z_3) = \alpha_3 z_2 z_3,$$

so that $\alpha_3 = 0$, giving $z_1^\tau \in \Bbbk^\times z_1$. Thus, $z_1 z_2^\tau = 0$, and, since $z_1 z_2 = 0$, this implies $z_2^\tau \in \Bbbk^\times z_2$, since the point scheme of A is the graph of an automorphism. Consequently,

$$z_1 z_3^\tau = z_3 z_1^\tau = \alpha_1 \mu_{13} z_1 z_3,$$

and so $z_1(z_3^\tau - \alpha_1 \mu_{13} z_3) = 0$. It follows that $z_3^\tau \in \Bbbk^\times z_3 + \Bbbk z_2$. Writing $z_3^\tau = \beta_1 z_3 + \beta_2 z_2$, where $\beta_1, \beta_2 \in \Bbbk$, we have

$$0 = (z_3^2)^\tau = (z_3^\tau)^2 = \beta_2 z_2(\beta_2 z_2 + 2\beta_1 z_3).$$

Thus, either $\beta_2 = 0$ or $\beta_2 z_2 + 2\beta_1 z_3 \in \Bbbk z_1$. Both cases imply $\beta_2 = 0$, so $z_3^\tau \in \Bbbk^\times z_3$. Similarly,

$$z_1 z_4^\tau = z_4 z_1^\tau = \alpha_1 \mu_{14} z_1 z_4 \qquad \text{and} \qquad z_3 z_4^\tau = z_4 z_3^\tau = \beta_1 \mu_{34} z_3 z_4,$$

which imply

$$z_1(z_4^\tau - \alpha_1 \mu_{14} z_4) = 0 = z_3(z_4^\tau - \beta_1 \mu_{34} z_4),$$

giving $z_4^\tau \in (\Bbbk z_2 + \Bbbk z_4) \cap (\Bbbk z_3 + \Bbbk z_4)$. Hence, $z_4^\tau \in \Bbbk^\times z_4$. Consequently, we may write $z_i^\tau = \lambda_i z_i$, for all i, where $\lambda_i \in \Bbbk^\times$ for all i, from which the equations $z_j z_i^\tau = z_i z_j^\tau$, for all i, j, together with the relations of B, imply that $\lambda_j = \mu_{ij} \lambda_i$ for all $i, j = 1, 3, 4$. Thus,

$$\mu_{41} \lambda_4 = \lambda_1 = \mu_{31} \lambda_3 = \mu_{31} \mu_{43} \lambda_4 = -\mu_{41} \lambda_4,$$

as $\mu_{13}\mu_{34} + \mu_{14} = 0$. Since $\mu_{14}\lambda_4 \neq 0$ and $\mathrm{char}(\Bbbk) \neq 2$, we have derived a contradiction, so the claim is proved.

We conclude the paper by listing some open problems.

1. In [16], a method was given for counting the number of points modules over a regular graded Clifford algebra when the point scheme is finite. It would be useful to find such an algorithm for regular graded skew Clifford algebras.
2. Regular graded skew Clifford algebras of global dimension four are elements of the scheme of quadratic algebras on four generators with six relations. It would be interesting to determine how many components of that scheme contain graded skew Clifford algebras, and if any of those components can be determined.
3. Is it possible to generalize the geometric methods in this paper to the setting where S is a regular quadratic algebra on n generators of global dimension n, but is not necessarily a skew polynomial ring? In particular, is the notion of a normalizing quadric system and the notion of base-point free useful for constructing new regular algebras in such a setting? Are there such geometric methods that can be applied to the situation in [9], in which if $\mathfrak{Q}$ is a regular normalizing sequence of length n of homogeneous elements of degree two in a regular quadratic algebra B on n generators, then the Koszul dual of $B/\langle\mathfrak{Q}\rangle$ is a regular quadratic algebra?

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