

SKEW CLIFFORD ALGEBRAS

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ABSTRACT. We introduce a generalization, called a *skew Clifford algebra*, of a Clifford algebra, and relate these new algebras to the notion of graded skew Clifford algebra that was defined in 2010. In particular, we examine homogenizations of skew Clifford algebras, and determine which skew Clifford algebras can be homogenized to create Artin-Schelter regular algebras. Just as (classical) Clifford algebras are the Poincaré-Birkhoff-Witt (PBW) deformations of exterior algebras, skew Clifford algebras are the \mathbb{Z}_2 -graded PBW deformations of quantum exterior algebras. We also determine the possible dimensions of skew Clifford algebras and provide several examples.

1. INTRODUCTION

An exterior algebra can be quantized, or deformed, to create a (classical) Clifford algebra. Indeed, Clifford algebras are precisely the set of Poincaré-Birkhoff-Witt (PBW) deformations of exterior algebras ([22]). An \mathbb{N} -graded analogue of a classical Clifford algebra is related to the geometric notion of complete intersection and is called a graded Clifford algebra in [14]; under certain geometric conditions, these algebras are quadratic Artin-Schelter regular (AS-regular) algebras. Although the classification of the AS-regular algebras of global dimension three is complete, the classification of the higher-dimensional cases is an open question that continues to generate considerable interest (see, for example, [11, 16, 21, 23]). Recently, in [8], it was shown that a skew (i.e., quantized) version of a graded Clifford algebra can be used to construct several new families of quadratic AS-regular algebras of global dimension four. Since most quadratic AS-regular algebras of global dimension three can be understood in the framework of graded skew Clifford algebras (cf. [18]), it is reasonable to hope that the skew-Clifford approach will be useful in understanding the higher-dimensional cases.

Our work herein is motivated by a desire to construct a skew version of the path from exterior algebras to graded Clifford algebras; to this end, we define a skew (i.e., quantized) version of a

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Clifford algebra that parallels the classical definition. We show that these algebras are precisely the \mathbb{Z}_2 -graded PBW deformations of quantum exterior algebras, thereby mirroring the connection between exterior algebras and classical Clifford algebras. We are also interested in knowing whether the relationship between classical Clifford algebras and graded Clifford algebras persists in the skew case. Can a skew Clifford algebra be homogenized to produce an \mathbb{N} -graded quadratic algebra? Is there a homogenization that produces a graded skew Clifford algebra? Does every graded skew Clifford algebra arise from such a homogenization? Which skew Clifford algebras can be homogenized to produce AS-regular algebras? Answering these questions could conceivably help with the general project of classifying AS-regular algebras. Moreover, given the valuable role that Clifford algebras play in physics and geometry, it is likely that skew Clifford algebras will eventually find applications in physics and noncommutative geometry.

This paper is organized as follows. In Section 2, we present the basic terminology and establish two equivalent ways of defining a skew Clifford algebra, parallel to the definitions of classical Clifford algebras. Section 3 investigates the possible vector-space dimensions of skew Clifford algebras, and includes a variety of illustrative examples. Section 4 presents skew Clifford algebras as PBW deformations of the quantum exterior algebras. In Section 5, we explore possible homogenizations of skew Clifford algebras, establish connections with the graded skew Clifford algebras introduced in [8], and determine which skew Clifford algebras can be homogenized to create AS-regular algebras. Section 6 compares our skew Clifford algebras with other generalizations of Clifford algebras found in the literature.

2. DEFINITIONS

In this section, we introduce the notion of a skew Clifford algebra in Definition 2.4, and determine a universal property for the algebra in Theorem 2.6. Our approach follows the classical setting, which we first recall.

Throughout the paper, \mathbb{k} denotes a field such that $\text{char}(\mathbb{k}) \neq 2$, and we denote the set of nonzero elements in \mathbb{k} by \mathbb{k}^\times . We first recall the definition of a finitely generated classical Clifford algebra via a universal property.

Definition 2.1. [12] Let V be a finite-dimensional vector space over \mathbb{k} together with a symmetric bilinear form, $\phi : V \times V \rightarrow \mathbb{k}$. A Clifford algebra associated with V and ϕ is an associative \mathbb{k} -algebra, $\text{Cl}(V, \phi)$, together with a linear map, $g : V \rightarrow \text{Cl}(V, \phi)$, satisfying the condition $g(v)^2 = \phi(v, v)$, for all $v \in V$, such that, for every \mathbb{k} -algebra A , and for every linear map $f : V \rightarrow A$ with

$$f(v)^2 = \phi(v, v) \cdot 1_A,$$

for all $v \in V$, there is a unique algebra homomorphism, $\bar{f} : \text{Cl}(V, \phi) \rightarrow A$, such that $f = \bar{f} \circ g$ as in the diagram:

$$\begin{array}{ccc} V & \xrightarrow{g} & \text{Cl}(V, \phi) \\ & \searrow f & \downarrow \bar{f} \\ & & A \end{array}$$

Since the universal property implies that $\text{Cl}(V, \phi)$ is unique (if it exists), one may, alternatively, define the classical Clifford algebra associated with V and ϕ via generators and relations as follows.

Definition 2.2. Let V be a vector space with basis $\{x_1, \dots, x_n\}$, and let $T(V)$ be the tensor algebra of V . The Clifford algebra associated with V and ϕ is the quotient of $T(V)$ by the ideal generated by all elements of the form $x_i \otimes x_j + x_j \otimes x_i - 2\phi(x_i, x_j) \cdot 1$ for all i, j .

In the literature, $\text{Cl}(V, \phi)$ is sometimes called a universal Clifford algebra.

In order to define the parallel notion of a quantized Clifford algebra, we need some additional terminology. We use $M(n, \mathbb{k})$ to denote the space of all $n \times n$ matrices with entries in \mathbb{k} .

Definition 2.3.

- (a) Following [6, Section I.2.1], $\mu = (\mu_{ij}) \in M(n, \mathbb{k})$ is called *multiplicatively antisymmetric* if $\mu_{ij}\mu_{ji} = \mu_{ii} = 1$ for all i, j .
- (b) Let $\mu \in M(n, \mathbb{k})$ be multiplicatively antisymmetric and let V be a vector space with basis $\mathcal{B} = \{x_1, \dots, x_n\}$. We call a bilinear form $\phi : V \times V \rightarrow \mathbb{k}$ *μ -symmetric* (relative to \mathcal{B}) if $\phi(x_i, x_j) = \mu_{ij}\phi(x_j, x_i)$ for all i, j .
- (c) [8] Let $\mu \in M(n, \mathbb{k})$ be multiplicatively antisymmetric. A matrix $M \in M(n, \mathbb{k})$ is called *μ -symmetric* if $M_{ij} = \mu_{ij}M_{ji}$ for all i, j .

If $\mu_{ij} = 1$ for all i, j , then μ -symmetric bilinear forms are in fact symmetric. Notice that the definition of μ -symmetry for a bilinear form is always relative to a specified basis. For this reason we will typically present ϕ via a μ -symmetric matrix $B \in M(n, \mathbb{k})$ that defines the values of ϕ on the relevant basis for V ; i.e., $B_{ij} = \phi(x_i, x_j)$ for all $i, j \in \{1, \dots, n\}$.

We would like to generalize Definitions 2.1 and 2.2 by allowing the symmetric bilinear form ϕ to be replaced with a μ -symmetric bilinear form. It is easy to state the μ -symmetric version of Definition 2.2, but one should note that such algebras can be trivial; i.e., of dimension zero (cf. Example 3.7).

Definition 2.4. Let V be a vector space with ordered basis $\{x_1, \dots, x_n\}$, and let ϕ be a μ -symmetric bilinear form associated with this basis. The skew Clifford algebra $\text{sCl}(V, \mu, \phi)$ associated with ϕ is the quotient of the tensor algebra $T(V)$ by the ideal generated by all elements of the form $x_i \otimes x_j + \mu_{ij}x_j \otimes x_i - 2\phi(x_i, x_j) \cdot 1$ for all i, j .

If $\mu_{ij} = 1$ for all i, j , then $\text{sCl}(V, \mu, \phi)$ is a classical Clifford algebra. If $\phi \equiv 0$, then $\text{sCl}(V, \mu, \phi)$ is a quantum exterior algebra (see [6]) which we denote by $\Lambda_\mu(V)$. We sometimes refer to $\text{sCl}(V, \mu, \phi)$ as the skew Clifford algebra associated to μ and B , where B is the μ -symmetric matrix determined by ϕ relative to the basis $\{x_1, \dots, x_n\}$.

Remark 2.5. Let R be the skew Clifford algebra associated to matrices μ and B . From the defining relations in Definition 2.4, we see that the opposite algebra R^{op} is the skew Clifford algebra associated with the transposed matrices μ^t and B^t .

The inclusion of V into $T(V)$, composed with the natural projection of $T(V)$ onto $\text{sCl}(V, \mu, \phi)$, creates a map which we call g . In the case of a (universal) Clifford algebra, g is necessarily injective,

$$\begin{array}{ccc} V & \hookrightarrow & T(V) \\ & \searrow g & \downarrow \pi \\ & & \text{sCl}(V, \mu, \phi) \end{array}$$

but this need not be true for skew Clifford algebras in general (cf. Example 3.7). We prove in Theorem 3.9 that the map $g : V \rightarrow \text{sCl}(V, \mu, \phi)$ is injective exactly when the skew Clifford algebra has dimension $2^{\dim(V)}$.

Theorem 2.6. *Let V be a vector space over \mathbb{k} with fixed ordered basis $\{x_1, \dots, x_n\}$ together with a μ -symmetric bilinear form, $\phi : V \times V \rightarrow \mathbb{k}$. Then $\text{sCl}(V, \mu, \phi)$ satisfies the condition*

$$g(x_i)g(x_j) + \mu_{ij}g(x_j)g(x_i) = 2\phi(x_i, x_j) \cdot 1$$

for all $i, j \in \{1, \dots, n\}$, and, for every \mathbb{k} -algebra A , and for every linear map $f : V \rightarrow A$, with

$$f(x_i)f(x_j) + \mu_{ij}f(x_j)f(x_i) = 2\phi(x_i, x_j) \cdot 1_A$$

for all $i, j \in \{1, \dots, n\}$, there is a unique algebra homomorphism, $\bar{f} : \text{sCl}(V, \mu, \phi) \rightarrow A$, such that $f = \bar{f} \circ g$ as in the diagram:

$$\begin{array}{ccc} V & \xrightarrow{g} & \text{sCl}(V, \mu, \phi) \\ & \searrow f & \vdots \bar{f} \\ & & A \end{array}$$

Proof. The defining relations of $\text{sCl}(V, \mu, \phi)$ guarantee that

$$g(x_i)g(x_j) + \mu_{ij}g(x_j)g(x_i) = 2\phi(x_i, x_j) \cdot 1,$$

for all $i, j \in \{1, \dots, n\}$. Let A be a \mathbb{k} -algebra and $f : V \rightarrow A$ be a linear map such that

$$f(x_i)f(x_j) + \mu_{ij}f(x_j)f(x_i) = 2\phi(x_i, x_j) \cdot 1_A,$$

for all i, j . Since the set $\{g(x_i)\}$ generates $\text{sCl}(V, \mu, \phi)$, it suffices to define \bar{f} on these generators. Set $\bar{f}(g(x_i)) := f(x_i)$ for all i . Since the $f(x_i)$ satisfy the same relations as the $g(x_i)$, \bar{f} is well defined. Uniqueness follows from the fact that $\{g(x_i)\}$ generates $\text{sCl}(V, \mu, \phi)$. \square

3. DIMENSION

In this section we explore the possible dimension of a skew Clifford algebra. A universal Clifford algebra on n generators has dimension 2^n . In contrast, a skew Clifford algebra on n generators can have much smaller dimension.

We use Bergman's Diamond Lemma, from [3], to calculate the dimension of skew Clifford algebras. Let $\text{sCl}(V, \mu, \phi)$ be a skew Clifford algebra as in Definition 2.4, and write $B_{ij} = \phi(x_i, x_j)$ for all $i, j \in \{1, \dots, n\}$, where $\{x_1, \dots, x_n\}$ is a fixed ordered basis of V . We order the generators so that $x_i < x_j$ if $i < j$. The defining relations for $\text{sCl}(V, \mu, \phi)$ provide initial ambiguities of the form $x_j^2 x_i$, $x_j x_i^2$ and $x_k x_j x_i$ for all $i < j < k$. These ambiguities are all resolvable provided the following linear equations hold for all distinct i, j and k :

$$(1) \quad 2(1 - \mu_{ij})B_{ij}x_i = (1 - \mu_{ij}^2)B_{ii}x_j,$$

$$(2) \quad 2(1 - \mu_{ij})B_{ij}x_j = (1 - \mu_{ij}^2)B_{jj}x_i,$$

$$(3) \quad (1 - \mu_{ij}\mu_{ik})B_{jk}x_i + (\mu_{ij} - \mu_{jk})B_{ik}x_j = (1 - \mu_{jk}\mu_{ik})B_{ij}x_k.$$

(One could use equation (3) with repeated indices to consolidate all three equations.) These equations hold in $\text{sCl}(V, \mu, \phi)$ without introducing new linear relations if and only if the coefficients in (1), (2) and (3) are all zero. This is equivalent to the following condition:

★ for all i and j , if $B_{ij} \neq 0$ then $\mu_{ik} = \mu_{kj}$ for all k (see Theorem 3.9).

Lemma 3.1. *The dimension of a skew Clifford algebra $\text{sCl}(V, \mu, \phi)$ is at most $2^{\dim(V)}$.*

Proof. Let $\{x_1, \dots, x_n\}$ be a basis for V . Since, in $\text{sCl}(V, \mu, \phi)$, we have

$$g(x_i)g(x_j) + \mu_{ij}g(x_j)g(x_i) = 2\phi(x_i, x_j) \cdot 1,$$

for all i, j , it is clear that 1 together with the products

$$(4) \quad g(x_{i_1})g(x_{i_2}) \cdots g(x_{i_k}), \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n,$$

span $\text{sCl}(V, \mu, \phi)$. This gives a spanning set with at most 2^n elements, and so the dimension of $\text{sCl}(V, \mu, \phi)$ is at most 2^n . \square

We say that a skew Clifford algebra $s\text{Cl}(V, \mu, \phi)$ has *full dimension* if its dimension is $2^{\dim(V)}$. We show, in Theorem 3.9, that this is equivalent to condition \star . We say that a skew Clifford algebra is *trivial* if its dimension is 0. Clifford algebras are examples of skew Clifford algebras with full dimension, but there exist many skew Clifford algebras of full dimension that are not Clifford algebras, as illustrated in Examples 3.3, 3.4, 3.5 and 3.8. Examples 3.6 and 3.11 demonstrate that dimensions strictly between zero and $2^{\dim(V)}$ can occur.

Remark 3.2. As a consequence of condition \star , if R is a skew Clifford algebra of full dimension such that $B \in M(n, \mathbb{k})$ is diagonal of rank n , then μ is a symmetric matrix with $\mu_{ij}^2 = 1$ for all i, j .

Example 3.3. For any $a \in \mathbb{k}^\times$ and $b \in \mathbb{k}$, let $\mu = \begin{pmatrix} 1 & a & 1 \\ a^{-1} & 1 & a \\ 1 & a^{-1} & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix}$. Up to isomorphism, we may take $b \in \{0, 1\}$. These algebras have dimension 8.

Example 3.4. If μ is any multiplicatively antisymmetric matrix and $B = 0$, then $s\text{Cl}(V, \mu, \phi) = \Lambda_\mu(V)$, the quantum exterior algebra. Quantum exterior algebras have full dimension.

Example 3.5. Let $\mu = (\mu_{ij})$ be a multiplicatively antisymmetric matrix such that $\mu_{ij} = -1$, for all $i \neq j$, and let B be any diagonal matrix. Up to isomorphism, we may take the diagonal entries in B to be either 0 or 1. These skew Clifford algebras of full dimension are quotients of polynomial rings.

Example 3.6. Let

$$\mu = \begin{pmatrix} 1 & \mu_{12} & \mu_{13} & 1 \\ \mu_{21} & 1 & \mu_{23} & 1 \\ \mu_{31} & \mu_{32} & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 & B_{14} \\ 0 & 0 & 0 & B_{24} \\ 0 & 0 & 0 & B_{34} \\ B_{14} & B_{24} & B_{34} & 1 \end{pmatrix},$$

where $B_{i4} \neq 0$, for all i . If $\mu_{23} = \mu_{13} = 1 \neq \mu_{12}$, then this algebra has dimension 8. If either μ_{13} or μ_{23} is not equal to 1, then the algebra has dimension 4.

Example 3.7. Let $\mu = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This algebra has dimension zero.

Example 3.8. Let $\mu = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, for some $a \in \mathbb{k}$. This algebra has dimension 8. We note that the algebra is not simple, since $x_3 + 1$ generates a proper ideal.

We note that, for generic data (μ, ϕ) , where μ is a multiplicatively antisymmetric matrix and ϕ is a μ -symmetric bilinear form, the associated skew Clifford algebra has dimension zero.

The Koszul dual, $\Lambda_\mu(V)^\dagger$, of $\Lambda_\mu(V)$ is a skew polynomial ring and the matrix B , in this case, defines a quadratic element $q \in \Lambda_\mu(V)^\dagger$ as follows. Using z_i to denote the generator in $\Lambda_\mu(V)^\dagger$ dual to x_i , and writing $z = (z_1, \dots, z_n)$, we define $q := zBz^\dagger$.

Theorem 3.9. *With notation as in Definition 2.4, let $\mathcal{B} = \{x_1, \dots, x_n\}$ and let B denote the matrix of ϕ with respect to \mathcal{B} . The following are equivalent:*

- (a) $g : V \rightarrow \text{sCl}(V, \mu, \phi)$ is injective,
- (b) $\text{sCl}(V, \mu, \phi)$ has dimension 2^n ,
- (c) the coefficients in equations (1), (2) and (3) are all zero,
- (d) condition \star holds,
- (e) in the skew polynomial ring $\Lambda_\mu(V)^!$, the element q determined by the matrix B is central.

Proof. (a) \Rightarrow (c). The set $\{g(x_1), \dots, g(x_n)\}$ spans $g(V)$ in $\text{sCl}(V, \mu, \phi)$. Any linear relation $\sum_i \alpha_i g(x_i) = 0$, $\alpha_i \in \mathbb{k}$, would imply that $\sum_i \alpha_i x_i \in \ker(g)$. Since g is injective, there can be no such nontrivial linear relation, and hence the coefficients in equations (1), (2) and (3) are all zero.

(c) \Rightarrow (b). Since the coefficients in equations (1), (2) and (3) are all zero, each of the $2^n - 1$ monomials in the set (4) is fully reduced, and hence the dimension of $\text{sCl}(V, \mu, \phi)$ is 2^n .

(b) \Rightarrow (a). If $\text{sCl}(V, \mu, \phi)$ has dimension 2^n , then the spanning set in the proof of Lemma 3.1 is a basis. This basis contains the set $\{g(x_1), \dots, g(x_n)\}$, and the linear independence of these elements implies that g is injective.

(c) \Leftrightarrow (d). If the coefficient on the right in equation (3) is zero for all i, j and k , then B_{ij} not equal to 0 implies that $\mu_{ik} = \mu_{jk}^{-1} = \mu_{kj}$, which is condition \star . If condition \star holds, then B_{ii} not equal to 0 implies that $\mu_{ij} = \mu_{ji} = \mu_{ij}^{-1}$, and B_{ij} not equal to 0 implies $\mu_{ij} = \mu_{jj} = 1$. Thus the coefficients in equation (1), (2) and (3) are all zero,

(d) \Leftrightarrow (e). If $B = 0$, then $q = 0$ and condition \star is vacuously true. Suppose $B \neq 0$. In particular, $B_{ij} \neq 0$ for some i, j , so the monomial $z_i z_j$ appears in the expression for q . It follows that q is central if and only if $z_i z_j$ is central for all i, j such that $B_{ij} \neq 0$. This happens if and only if, for each k , we have $(z_i z_j) z_k = z_k (z_i z_j) = \mu_{ik} \mu_{jk} z_i z_j z_k$ for all i, j such that $B_{ij} \neq 0$; that is, for each k , we have $\mu_{ik} = \mu_{kj}$ whenever $B_{ij} \neq 0$. This is condition \star . \square

Corollary 3.10. *With notation as in Theorem 3.9, if $\text{sCl}(V, \mu, \phi)$ has full dimension, then B is a symmetric matrix.*

Proof. By Theorem 3.9, the hypothesis implies \star holds, which implies that $\mu_{ij} = \mu_{jj} = 1$ whenever $B_{ij} \neq 0$. Hence, $B_{ji} = B_{ij}$ for all i, j . \square

The following example shows that the converse to Corollary 3.10 does not hold.

Example 3.11. Suppose $\text{char}(\mathbb{k}) \notin \{2, 3\}$, and let $\mu = \begin{pmatrix} 1 & 2 \\ 1/2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Here B is symmetric, but the algebra has dimension 2, since the relations $x_1 x_2 = -2x_2 x_1$ and $x_2^2 = 1$ imply that $x_1 = 0$.

Theorem 3.12. *With notation as in Definition 2.4 and Theorem 3.9, if $\text{sCl}(V, \mu, \phi)$ is nontrivial, then $\text{sCl}(V, \mu, \phi)$ is isomorphic to $\text{sCl}(V', \mu', \phi|_{V'})$, where μ' is the multiplicatively antisymmetric submatrix of μ corresponding to a subset \mathcal{B}' of \mathcal{B} , V' is spanned by \mathcal{B}' , and $g|_{V'} : V' \rightarrow \text{sCl}(V, \mu, \phi)$ is injective.*

Proof. If the coefficients in equations (1), (2) and (3) are all zero, then by Theorem 3.9, g is injective and we take $V' = V$ and $\mu' = \mu$.

Assume from now on that the coefficients in equations (1), (2) and (3) are not all zero. We proceed by induction on the size of \mathcal{B} . If $|\mathcal{B}| = 1$, then $\text{sCl}(V, \mu, \phi) = \mathbb{k}[x]/\langle x^2 - b \rangle$ for some $b \in \mathbb{k}$. This algebra has basis $\{1, x\}$ and dimension 2, and clearly g is injective.

Now let $\mathcal{B} = \{x_1, \dots, x_n\}$ where $n > 1$. Assume inductively that a skew Clifford algebra associated to a vector space of dimension less than n has dimension 2^l for some $l < n$ and is isomorphic to a skew Clifford algebra associated with the span of a subset of \mathcal{B} on which g is injective. Since equations (1), (2) and (3) include at least one nonzero coefficient, there are three possible consequences in $\text{sCl}(V, \mu, \phi)$:

- (i) $x_j = 0$ for some j ;
- (ii) $x_k \neq 0$ for all k , but $x_j \in \mathbb{k}^\times x_i$ for some $i \neq j$;
- (iii) (i) and (ii) are false and equation (3) implies $x_k \in \mathbb{k}^\times x_i + \mathbb{k}^\times x_j$ for some i, j, k .

(i) First we consider the possibility that $x_j = 0$ in $\text{sCl}(V, \mu, \phi)$ for some j . Since $\text{sCl}(V, \mu, \phi)$ is nontrivial, each relation of the form $x_j x_m + \mu_{jm} x_m x_j = 2B_{jm}$ vanishes. It follows that $B_{jm} = 0 = B_{mj}$ for all m . Hence, x_j plays no role in $\text{sCl}(V, \mu, \phi)$, so we may replace \mathcal{B} with the subset $\mathcal{B} \setminus \{x_j\}$. The induction hypothesis implies that the algebra has dimension 2^l for some $l \leq n - 1$ and is equal to $\text{sCl}(V', \mu', \phi|_{V'})$ where $g|_{V'}$ is injective and V' is spanned by a subset of $\mathcal{B} \setminus \{x_j\}$.

(ii) Next we assume (ii) holds. Suppose $x_j = \lambda x_i$ where $\lambda \in \mathbb{k}^\times$. In particular, there must either be a pair of subscripts i, j such that equation (1) or (2) has nonzero coefficients, or there is a triple of subscripts i, j, p such that exactly two of the coefficients in equation (3) are nonzero. If $x_k \in \mathbb{k}^\times x_i$ for all k , then there exists $b \in \mathbb{k}$ such that $\text{sCl}(V, \mu, \phi) \cong \mathbb{k}[x]/\langle x^2 - b \rangle$, which has dimension 2 and the result follows. So we assume that there exists k such that $x_k \notin \mathbb{k}x_i$. We also assume, without loss of generality, that $j > i$. The relation $x_j = \lambda x_i$ introduces additional ambiguities to be resolved via Bergman's Diamond Lemma; namely, $x_j x_i$, x_j^2 , $x_j x_k$, for all $k < j$, and $x_k x_j$ for all $k > j$. Any relations produced, by resolving these ambiguities, that involve only scalars are resolvable, since the algebra is nontrivial. The other relations produced by resolving these ambiguities have the form

$$(5) \quad (\mu_{jk} - \mu_{ik})\lambda x_k x_i = 2(B_{jk} - \lambda B_{ik}),$$

for all $k < j$, where $x_k \notin \mathbb{k}x_i$, and

$$(6) \quad (\mu_{kj} - \mu_{ki})\lambda x_i x_k = 2(B_{kj} - \lambda B_{ki}),$$

for all $k > j$, where $x_k \notin \mathbb{k}x_i$. Multiplying (5) on the left by x_k yields $(\mu_{jk} - \mu_{ik})\lambda B_{kk}x_i = 2(B_{jk} - \lambda B_{ik})x_k$. As $x_k \notin \mathbb{k}x_i$, it follows that

$$(7) \quad B_{jk} = \lambda B_{ik}$$

and that $B_{kk}(\mu_{jk} - \mu_{ik}) = 0$, for all $k < j$ such that $x_k \notin \mathbb{k}x_i$. Similarly, multiplying (5) on the right by x_j (respectively, x_i) implies that $B_{jj}(\mu_{jk} - \mu_{ik}) = 0 = B_{ii}(\mu_{jk} - \mu_{ik})$ for all $k < j$ such that $x_k \notin \mathbb{k}x_i$. Moreover, similar computations using (6) yield analogous formulae for all $k > j$ such that $x_k \notin \mathbb{k}x_i$.

Hence, if $\mu_{jk} = \mu_{ik}$ for all k with the property that $x_k \notin \mathbb{k}x_i$, then all the ambiguities that involve x_j are resolvable, which means that the relation $x_j = \lambda x_i$ contributes no additional relations. In this case, it then follows that x_j is redundant in $\text{sCl}(V, \mu, \phi)$, and so the result follows as in (i). Thus, in order to complete the proof of case (ii), it suffices to prove that $\mu_{jk} = \mu_{ik}$ if $x_k \notin \mathbb{k}x_i$.

Suppose, for a contradiction, that $x_k \notin \mathbb{k}x_i$, $k < j$ and $\mu_{jk} \neq \mu_{ik}$. It follows that $0 = B_{ii} = B_{jj} = B_{kk}$. Since $B_{ii} = 0$, we obtain

$$2B_{ij} = x_i x_j + \mu_{ij} x_j x_i = \lambda(1 + \mu_{ij})x_i^2 = \lambda(1 + \mu_{ij})B_{ii} = 0.$$

Considering equation (1) for the pairs (i, k) and (j, k) , and noting that the right side of the equation is zero in each case, we find that $(1 - \mu_{ik})B_{ik} = 0 = (1 - \mu_{jk})B_{jk}$. As μ_{ik} and μ_{jk} cannot both be equal to 1, we obtain that B_{jk} or B_{ik} is zero, and so, by (7), both are zero. This means that $B_{cd} = 0$ for all $c, d \in \{i, j, k\}$. Recall that case (ii) occurs because either equation (1) or (2) has nonzero coefficients for subscripts i, j , or there is a triple of subscripts i, j, p such that exactly two of the coefficients in equation (3) are nonzero. Since $B_{cd} = 0$ for all $c, d \in \{i, j, k\}$, equations (1) and (2) have zero coefficients for subscripts i, j . It follows that, in order to have $x_j \in \mathbb{k}^\times x_i$, there exists $p \notin \{i, j, k\}$ such that $B_{ip} \neq 0 \neq B_{jp}$. Considering equation (1) with the subscript pairs (i, p) and (j, p) , and noting that the right side of the equation is zero in each case, we find that $\mu_{ip} = 1 = \mu_{jp}$. Similarly, considering equation (3) for the subscript triple (i, p, k) , and noting that the middle term vanishes since $B_{ik} = 0$, and using the assumption that $x_k \notin \mathbb{k}x_i$ and the fact that $B_{ip} \neq 0$, we obtain $1 - \mu_{pk}\mu_{ik} = 0$. However, a similar discussion using equation (3) with the subscript triple (p, j, k) implies that $1 - \mu_{pk}\mu_{jk} = 0$, which is a contradiction. Similarly, a contradiction is obtained if one assumes $k > j$ instead.

(iii) Lastly, we assume that (i) and (ii) are false and that equation (3) implies $x_k \in \mathbb{k}^\times x_i + \mathbb{k}^\times x_j$ for some i, j, k . In particular, all three coefficients in equation (3) are nonzero for the subscript triple (i, j, k) , and all the coefficients in equations (1) and (2) are zero. Applying this to the

subscript pairs (i, j) and (j, k) , we obtain $B_{ij}(\mu_{ij} - 1) = 0 = B_{jk}(\mu_{jk} - 1)$. It follows that at least one of the coefficients in equation (3) is zero, which is a contradiction. \square

Corollary 3.13. *With notation as in Definition 2.4, if $\text{sCl}(V, \mu, \phi) \neq \{0\}$, then $\dim(\text{sCl}(V, \mu, \phi)) = 2^j$, where $1 \leq j \leq \dim(V)$.*

Proof. By Theorem 3.12, the map g is injective on a subspace V' of V . Let $j = \dim(V')$ and apply Theorem 3.9. \square

Since, by Theorem 3.12, any nontrivial skew Clifford algebra can be identified with a skew Clifford algebra of full dimension, we will often find it convenient to assume that the skew Clifford algebra under discussion has full dimension. When we do so, Corollary 3.10 implies that we may also assume that ϕ is a symmetric bilinear form, and we can define the corresponding quadratic form on V by $\Phi(x) := \phi(x, x)$ for all $x \in V$.

4. GRADINGS AND DEFORMATIONS

Skew Clifford algebras have a \mathbb{Z}_2 -grading which is obtained by assigning each x_i an odd degree, and this makes each skew Clifford algebra a superalgebra that can be decomposed into the direct sum of its odd and even parts. As with classical Clifford algebras, the even part forms a subalgebra for which the odd part is a module. However, skew Clifford algebras are typically not supercommutative.

Classical Clifford algebras are quantizations of the exterior algebra in that, for a given vector space V , each classical Clifford algebra is isomorphic as a vector space to the exterior algebra on V , although the two algebras have different multiplications. As a consequence of Theorem 3.12, each nontrivial skew Clifford algebra $\text{sCl}(V, \mu, \phi)$ is a quantization of the quantum exterior algebra $\Lambda_{\mu'}(V')$, where V' is a subspace of V and μ' is the corresponding multiplicatively antisymmetric submatrix of μ .

Any skew Clifford algebra $R = \text{sCl}(V, \mu, \phi)$ is also a nonhomogeneous quadratic algebra in the sense of Braverman and Gaitsgory in [5], which is obtained by deforming the relations of the associated quantum exterior algebra. Specifically, consider the natural filtration $F^i T = \{\oplus T^j(V) : j \leq i\}$ on the tensor algebra $T(V)$, and let $\pi : F^2 T \rightarrow V \otimes V$ denote the projection onto $V \otimes V$. If $P \subset F^2 T$ is a subspace such that $R = T(V)/\langle P \rangle$, then the quantum exterior algebra $\Lambda_{\mu}(V)$ can be represented as $T(V)/\langle \pi(P) \rangle$, and every \mathbb{Z}_2 -graded nonhomogeneous quadratic algebra obtained by deforming the relations of $\Lambda_{\mu}(V)$ is a skew Clifford algebra.

Moreover, R inherits a filtration from $T(V)$, and so we can consider the associated graded algebra $\text{gr}(R)$. Following [5], R is called a PBW deformation of $\Lambda_{\mu}(V)$ if the natural projection

$\Lambda_\mu(V) \rightarrow \text{gr}(R)$ is an isomorphism. In the case of classical Clifford algebras, the associated graded algebra is isomorphic to the exterior algebra, making every Clifford algebra a PBW deformation of the corresponding exterior algebra. In fact, classical Clifford algebras are exactly the set of PBW deformations of exterior algebras ([22]). It is natural to ask whether skew Clifford algebras are PBW deformations of quantum exterior algebras; i.e., is the associated graded algebra of a skew Clifford algebra the same as the quantum exterior algebra obtained by replacing ϕ with 0? The next result shows that every nontrivial skew Clifford algebra is a PBW deformation of some quantum exterior algebra.

Proposition 4.1. *With notation as in Definition 2.4, let $\text{sCl}(V, \mu, \phi)$ be nontrivial. Any \mathbb{Z}_2 -graded PBW deformation of the quantum exterior algebra $\Lambda_\mu(V)$ is a non-trivial skew Clifford algebra, and $\text{sCl}(V, \mu, \phi)$ is a PBW deformation of a quantum exterior algebra $\Lambda_{\mu'}(V')$, where V' is a subspace of V and μ' is the corresponding multiplicatively antisymmetric submatrix of μ .*

Proof. Any \mathbb{Z}_2 -graded deformation $T(V)/\langle P \rangle$ of a quantum exterior algebra $T(V)/\langle \pi(P) \rangle$ is a skew Clifford algebra. If $T(V)/\langle P \rangle$ is a PBW deformation, then $\text{gr}(T(V)/\langle P \rangle)$ and $T(V)/\langle \pi(P) \rangle$ have the same dimension, and hence $T(V)/\langle P \rangle$ has full dimension.

By Theorem 3.12, $\text{sCl}(V, \mu, \phi) \cong \text{sCl}(V', \mu', \phi')$, where μ' is the multiplicatively antisymmetric submatrix of μ corresponding to the subspace V' of V , ϕ' is the corresponding restriction of ϕ , and $\text{sCl}(V', \mu', \phi')$ has full dimension. The canonical map from the quantum exterior algebra $\Lambda_\mu(V')$ to $\text{gr}(\text{sCl}(V', \mu', \phi'))$ is always onto. The algebras $\text{sCl}(V', \mu', \phi')$ and $\text{gr}(\text{sCl}(V', \mu', \phi'))$ have the same dimension, and since $\text{sCl}(V', \mu', \phi')$ has full dimension, $\text{gr}(\text{sCl}(V', \mu', \phi'))$ and $\Lambda_\mu(V')$ are equidimensional. Thus the canonical map is an isomorphism and $\text{sCl}(V', \mu', \phi')$ is a PBW deformation of $\Lambda_\mu(V')$.

Thus, non-trivial skew Clifford algebras are exactly the set of \mathbb{Z}_2 -graded PBW deformations of quantum exterior algebras. \square

5. HOMOGENIZATIONS

In this section we show how skew Clifford algebras can be homogenized to create \mathbb{N} -graded algebras, and we identify the skew Clifford algebras that correspond to \mathbb{N} -graded algebras with good homological properties.

A skew Clifford algebra $R = \text{sCl}(V, \mu, \phi)$ of full dimension can be homogenized to create an \mathbb{N} -graded algebra A by adjoining a single degree-two central generator y . We set

$$A := \frac{\mathbb{k}\langle X_1, \dots, X_n \rangle[y]}{\langle X_i X_j + \mu_{ij} X_j X_i - 2B_{ij} y : 1 \leq i, j \leq n \rangle},$$

where the square brackets indicate that y is central in this algebra. Clearly $A/(y-1)A \cong R$ and $A/yA \cong \Lambda_\mu(V)$. By Proposition 4.1, $\text{gr}(R) \cong \Lambda_\mu(V)$, and so, by [7, Theorem 1.3], y is a regular element in A . It is also clear that if the matrix B is not identically zero, then y is a redundant generator, so that A is in fact generated by degree-one elements. In this case, it turns out that the centrality of y is a consequence of the degree-two relations, as the next theorem shows.

Theorem 5.1. *If R has full dimension and $B \neq 0$, then A is a quadratic algebra.*

Proof. We show that the quadratic relations in A imply that y is central. Firstly, suppose $B_{ii} \neq 0$ for some i . Without loss of generality, take $i = 1$, so $y = X_1^2$ in A . Since R has full dimension, condition \star implies that $\mu_{1k}^2 = 1$ for all k . It follows that $X_k y = y X_k + 2B_{k1}(1 - \mu_{k1})X_1^3$ for all k . However, condition \star implies that $B_{k1}(1 - \mu_{k1}) = 0$, and so $X_k y = y X_k$ for all k .

Now suppose $B_{ii} = 0$ for all i . Since B is nonzero, we may assume, without loss of generality, that $2B_{12} = 1$, so that $\mu_{12} = 1$ (by \star) and $y = X_1 X_2 + X_2 X_1$ in A . If $B_{k1} \neq 0$ for some $k \neq 1$, then $2B_{k1}y = X_1 X_k + X_k X_1$, so that $X_k y = (2B_{k1})^{-1} X_k (X_1 X_k + X_k X_1) = (2B_{k1})^{-1} X_k X_1 X_k = (2B_{k1})^{-1} (X_1 X_k + X_k X_1) X_k = y X_k$. Similarly, if $B_{k2} \neq 0$ for some $k \neq 2$, then X_k commutes with y . If $B_{k1} = 0 = B_{k2}$ for some $k \geq 3$, then $X_k y = \mu_{k1} \mu_{k2} y X_k$, and since $B_{12} \neq 0$, condition \star implies that $\mu_{k1} \mu_{k2} = 1$. \square

To relate skew Clifford algebras of full dimension to the graded skew Clifford algebras studied in [8], we construct a nonstandard homogenization of R by adjoining n degree-two central elements y_1, \dots, y_n . This new algebra will depend on a choice of μ -symmetric matrices $M_1, \dots, M_n \in M(n, \mathbb{k})$ that satisfy $\sum_{k=1}^n (M_k)_{ij} = 2B_{ij}$ for all i, j . We define the \mathbb{N} -graded \mathbb{k} -algebra $A(n)$ as

$$(8) \quad A(n) := \frac{\mathbb{k}\langle X_1, \dots, X_n \rangle [y_1, \dots, y_n]}{\langle X_i X_j + \mu_{ij} X_j X_i - \sum_{k=1}^n (M_k)_{ij} y_k : 1 \leq i, j \leq n \rangle},$$

i.e. $A(n)$ has $n(3n-1)/2$ relations that make y_k central in $A(n)$ for all k , and another $(n^2+n)/2$ relations derived from the relations in R . The algebra R is a “dehomogenization” of $A(n)$ in that $A(n)/\langle y_1 - 1, \dots, y_n - 1 \rangle \cong R$. The algebra $A(n)$ is an *iterated central extension* of $\Lambda_\mu(V)$ in that each y_k is central in $A(n)$ and $A(n)/\langle y_1, \dots, y_n \rangle \cong \Lambda_\mu(V)$. Note that, by construction, X_j^2 is

$$\begin{array}{ccc} & & R \\ & \nearrow & \\ A(n) & \longrightarrow & A \\ & \searrow & \\ & & \Lambda_\mu(V) \end{array}$$

central in $A(n)$ for all j . Moreover, $A(n)$ maps onto A , and, as is the case with A , it is possible for $A(n)$ to be a quadratic algebra generated by degree-one elements; Theorems 5.4 and 5.5 address this possibility.

The algebra $A(n)$ is a graded skew Clifford algebra, as defined in [8].

Definition 5.2. [8] Fix μ -symmetric matrices $M_1, \dots, M_n \in M(n, \mathbb{k})$. A *graded skew Clifford algebra* associated to μ and M_1, \dots, M_n is a graded \mathbb{k} -algebra on degree-one generators x_1, \dots, x_n and on degree-two generators y_1, \dots, y_n , with defining relations as follows:

- (a) degree-two relations of the form $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^n (M_k)_{ij} y_k$ for all $i, j \in \{1, \dots, n\}$, and
- (b) degree-three and degree-four relations (possibly obtainable from the quadratic relations in (a)) that guarantee the existence of a normalizing sequence $\{r_1, \dots, r_n\}$ of homogeneous elements of degree two that span $\mathbb{k}y_1 + \dots + \mathbb{k}y_n$.

The notion of graded skew Clifford algebra is a generalization of the notion of graded Clifford algebra (cf. [8, 14]).

We now show that if $\phi \neq 0$, then the skew Clifford algebra R is the image of a graded skew Clifford algebra that is generated by degree-one elements. Consider a skew Clifford algebra $\text{sCl}(V, \mu, \phi)$, where V has basis $\mathcal{B} = \{x_1, \dots, x_n\}$ and μ -symmetric bilinear form ϕ with respect to \mathcal{B} , and let $D = (D_{ij})$ be the μ -symmetric matrix given by $D_{ij} = 2\phi(x_i, x_j)$.

Lemma 5.3. *Suppose $\mu \in M(n, \mathbb{k})$ is multiplicatively antisymmetric and $D \in M(n, \mathbb{k})$ is μ -symmetric. If $D \neq 0$, then $D = \sum_{i=1}^n M_i$, where M_1, \dots, M_n are linearly independent μ -symmetric matrices.*

Proof. When creating μ -symmetric matrices, we have complete freedom to select entries on and above the main diagonal. The entries below the main diagonal are then determined by μ . Consequently, the vector space of μ -symmetric $n \times n$ matrices has dimension $n(n+1)/2$. Since $n \leq n(n+1)/2$, the nonzero μ -symmetric matrix D can be written as a sum of n linearly independent μ -symmetric matrices. \square

Theorem 5.4. *With notation as in Definition 2.4, if $\phi \neq 0$, then $\text{sCl}(V, \mu, \phi)$ is a quotient of a graded skew Clifford algebra that is generated by degree-one elements.*

Proof. Let $\mathcal{B} = \{x_1, \dots, x_n\}$ be a fixed ordered basis of V , and let B be the matrix of ϕ with respect to \mathcal{B} . As a consequence of Lemma 5.3, we can choose a partition of $2B$ into linearly independent μ -symmetric matrices M_1, \dots, M_n . We use these matrices, M_1, \dots, M_n , to define a graded skew Clifford algebra $A(n)$ as in (8). By construction, $A(n)/\langle y_1 - 1, \dots, y_n - 1 \rangle \cong \text{sCl}(V, \mu, \phi)$. Since M_1, \dots, M_n are linearly independent, [8, Lemma 1.13] implies that $A(n)$ is generated by X_1, \dots, X_n . \square

Thus, every skew Clifford algebra $R = \text{sCl}(V, \mu, \phi)$, with nonzero ϕ , can be homogenized (in the above manner) to create a graded skew Clifford algebra that is generated by degree-one elements.

The converse is true for (classical) graded Clifford algebras: every graded Clifford algebra is a nonstandard homogenization of a classical Clifford algebra; but this converse does not hold in general for their skew analogues (as defined in this article). In particular, the graded algebra produced by this homogenization contains central elements of degree two given by the y_k . Since graded skew Clifford algebras need not have central elements of degree two, not every graded skew Clifford algebra occurs as a homogenization of a skew Clifford algebra.

The homogenized algebra $A(n)$ maps onto R and is generated by the pre-images of the generators of R . Although the linear independence of the matrices M_1, \dots, M_n guarantees, via Theorem 5.4, that $A(n)$ can be generated by degree-one elements, $A(n)$ might not be quadratic. Nevertheless, $A(n)$ is noetherian, and although it is noncommutative, it can have some other nice properties of the commutative polynomial ring on n variables, such as being Auslander regular ([4]), Artin-Schelter regular ([1]), or a domain. As discussed in [8], if the graded skew Clifford algebra $A(n)$ is Auslander regular, then it is also Artin-Schelter regular and a domain. Since the family \mathcal{F} of graded skew Clifford algebras includes many Artin-Schelter regular algebras (cf. [18]), it is natural to ask which skew Clifford algebras are quotients of quadratic regular algebras in \mathcal{F} ; this issue is addressed in our final result.

Theorem 5.5. *With notation as in Definition 2.4, let $R = \text{sCl}(V, \mu, \phi)$, where $\phi \neq 0$ and $\dim(V) = n < \infty$. If R has full dimension, then we can choose μ -symmetric matrices $M_1, \dots, M_n \in M(n, \mathbb{k})$ such that the graded skew Clifford algebra $A(n)$ associated to μ and M_1, \dots, M_n is a quadratic Auslander-regular algebra if and only if $\mu_{ij}^2 = 1$ for all i, j .*

Proof. Suppose R has full dimension. As before, we write $B_{ij} = \phi(x_i, x_j)$. Suppose first that $A(n)$ is quadratic and Auslander regular. If $B_{ij} \neq 0$, then, by condition \star , we have $\mu_{ik} = \mu_{kj}$ for every k . In particular, by taking $k = j$, we obtain $\mu_{ij} = \mu_{jj} = 1$. On the other hand, if $B_{ij} = 0$, consider $X_i X_j + \mu_{ij} X_j X_i = \sum_k (M_k)_{ij} y_k$. For convenience, let W denote $\sum_k (M_k)_{ij} y_k$, and note that the image of W in R is zero. Since W and X_j^2 are central in $A(n)$, we have $X_j^2 X_i = X_i X_j^2 = \mu_{ij}^2 X_j^2 X_i + X_j(1 - \mu_{ij})W$. Since $A(n)$ is a domain, $X_j X_i = \mu_{ij}^2 X_j X_i + (1 - \mu_{ij})W$, and thus $(1 - \mu_{ij}^2)X_j X_i = (1 - \mu_{ij})W$. Projecting this equation onto R gives $(1 - \mu_{ij}^2)x_j x_i = 0$, but, since R has full dimension, $x_j x_i \neq 0$ in R , so $\mu_{ij}^2 = 1$.

For the converse, suppose that $\mu_{ij}^2 = 1$ for all i, j . If $\dim(V) = 1$, then $R \cong \mathbb{k}[x]/\langle x^2 - 1 \rangle$ and $\Lambda_\mu(V) \cong \mathbb{k}[x]/\langle x^2 \rangle$. So, in this case, $A(n) = \mathbb{k}[x]$ and is an Auslander-regular graded skew Clifford algebra. From now on, assume that $\dim(V) > 1$. We will construct a homogenization $A(n)$ (as defined earlier in this section) of R that is Auslander regular and generated by elements of degree one.

Consider first the case that B is diagonal, so $B_{ij} = 0$ for all $i \neq j$. Since $B \neq 0$, we may assume, without loss of generality, that $B_{11} \neq 0$. We define diagonal matrices M_1, \dots, M_n by

$$(M_k)_{ij} = \begin{cases} 2B_{kk} & \text{if } k = i = j \text{ and } B_{kk} \neq 0, \\ -1 & \text{if } 1 \neq k = i = j \text{ and } B_{kk} = 0, \\ 1 & \text{if } 1 = k < i = j \text{ and } B_{jj} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\sum_{k=1}^n (M_k)_{ij} = 2B_{ij}$ and that M_1, \dots, M_n are linearly independent, so that the y_k are redundant generators for $A(n)$. Moreover, since $\mu_{ij}^2 = 1$ and each M_k is diagonal, the quadratic relations imply that $X_i X_j^2 = X_j^2 X_i$ for all i, j , giving that X_j^2 is central for all j . Furthermore, in $A(n)$, we have

$$y_j = \begin{cases} X_j^2/B_{jj} & \text{if } B_{jj} \neq 0, \\ (X_1^2/B_{11}) - 2X_j^2 & \text{if } B_{jj} = 0, \end{cases}$$

so y_j is central for all j . It follows that, in this case, $A(n)$ is the skew polynomial ring

$$\mathbb{k}\langle X_1, \dots, X_n \rangle / \langle X_i X_j + \mu_{ij} X_j X_i : 1 \leq i < j \leq n \rangle,$$

which is a quadratic Auslander-regular algebra.

Now suppose that B is not diagonal. Since $B \neq 0$, by reordering the x_i and scaling, we may assume $B_{12} = 1$. Since R has full dimension, condition \star implies that $\mu_{12} = 1$. Similarly, if $B_{11}B_{22} = 0$, we may assume $B_{22} = 0$. We define μ -symmetric matrices M_1, \dots, M_n as follows:

$$(M_1)_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \text{ and } B_{11} = 0, \\ -1 & \text{if } i = j \geq 2 \text{ and } B_{jj} = 0, \\ 0 & \text{if } i = j \geq 2 \text{ and } B_{jj} \neq 0, \\ 2B_{ij} & \text{otherwise,} \end{cases} \quad (M_k)_{ij} = \begin{cases} 1 & \text{if } i = j = k \text{ and } B_{kk} = 0, \\ 2B_{kk} & \text{if } i = j = k \text{ and } B_{kk} \neq 0, \\ -1 & \text{if } k = 2, i = j = 1 \text{ and } B_{11} = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $k \geq 2$.

The matrices M_k satisfy $\sum_{k=1}^n (M_k)_{ij} = 2B_{ij}$ and are μ -symmetric and linearly independent since $2B_{12} \neq 0$. Moreover, $A(n)/\langle y_1 - 1, \dots, y_n - 1 \rangle \cong R$ and $A(n)/\langle y_1, \dots, y_n \rangle \cong \Lambda_\mu(V)$. We now show that the centrality of the y_i is a consequence of the quadratic relations, so that $A(n)$ is in fact a quadratic algebra.

We first consider y_1 . If $B_{11} = 0 = B_{22}$, then $2X_1^2 = y_1 - y_2$, $X_1X_2 + X_2X_1 = 2y_1$ and $2X_2^2 = -y_1 + y_2$. In particular, $X_1^2 = -X_2^2$, which implies that y_1 commutes with X_1 and X_2 . However, if $B_{11}B_{22} \neq 0$, then $X_1^2 = B_{11}y_1$, $X_1X_2 + X_2X_1 = 2y_1$ and $X_2^2 = B_{22}y_2$. In this case, we have $y_1X_1 = X_1y_1$ and

$$B_{11}y_1X_2 = X_1^2X_2 = 2X_1y_1 - X_1X_2X_1 = X_2X_1^2 = B_{11}X_2y_1,$$

so, again, y_1 commutes with X_1 and X_2 . On the other hand, if $B_{11}B_{22} = 0$ and $(B_{11}, B_{22}) \neq (0, 0)$, we may assume, as before, that $B_{22} = 0 \neq B_{11}$. In this case, $X_1^2 = B_{11}y_1$, $X_1X_2 + X_2X_1 = 2y_1$

and $2X_2^2 = y_2 - y_1$. Thus $y_1X_1 = X_1y_1$. Moreover, resolving X_1^3 implies that $X_1^2X_2 = X_2X_1^2$, from which we find $y_1X_2 = X_2y_1$. Hence, in all three scenarios, y_1 commutes with X_k for $k = 1, 2$.

For $k > 2$, we use the quadratic relations $X_kX_i + \mu_{ki}X_iX_k = 2B_{ki}y_1$, $i = 1, 2$, together with $2y_1 = X_1X_2 + X_2X_1$, to calculate

$$X_ky_1 = \mu_{k1}\mu_{k2}y_1X_k + B_{k1}(1 - \mu_{k2})y_1X_2 + B_{k2}(1 - \mu_{k1})y_1X_1.$$

As $B_{12} \neq 0$, condition \star implies that $\mu_{k1}\mu_{k2} = 1$. If $B_{k1} \neq 0$, then condition \star implies that $\mu_{k2} = \mu_{21} = 1$, and similarly if $B_{k2} \neq 0$ then $\mu_{k1} = 1$. Thus, y_1 commutes with X_k for all k .

Now observe that, for $j > 1$, we have

$$y_j = \begin{cases} X_j^2/B_{jj} & \text{if } B_{jj} \neq 0, \\ 2X_j^2 + y_1 & \text{if } B_{jj} = 0. \end{cases}$$

Thus, for $j > 1$, y_j commutes with X_k , for all k , provided that X_j^2 commutes with X_k , for all k . One may verify this is indeed the case by subtracting $(X_kX_j + \mu_{kj}X_jX_k = 2B_{jk}y_1)X_j$ from $X_j(X_jX_k + \mu_{jk}X_kX_j = 2B_{kj}y_1)$, and applying condition \star and our hypothesis that $\mu_{jk}^2 = 1$ for all j, k . Lastly, since each y_j commutes with each X_k and since y_j is a linear combination of products of the form X_aX_b , the y_j commute with each other. It follows that the centrality of the y_k in $A(n)$ is a consequence of only the degree-two relations of $A(n)$, and so $A(n)$ is quadratic.

We now show that $A(n)$ is an Auslander-regular algebra by examining two cases. Without loss of generality, we may assume that either $B_{11} = B_{22} = 0$ or, by reversing the roles of x_1 and x_2 if necessary, that $B_{11} \neq 0$.

Case I: $B_{11} = B_{22} = 0$. Given that $A(n)$ is quadratic and given the definition of M_1, \dots, M_n , the defining relations of $A(n)$ are of the form: $2X_1^2 = y_1 - y_2$, $2X_2^2 = y_2 - y_1$, $2X_k^2 \in \mathbb{k}y_1 + \mathbb{k}^\times y_k$, $X_iX_j + \mu_{ij}X_jX_i = 2B_{ij}y_1$, for all $k \geq 3$, and $1 \leq i \neq j \leq n$. In particular, as $B_{12} = 1$, we have $y_1 = \frac{1}{2}(X_1X_2 + X_2X_1)$, $y_2 = 2X_2^2 + \frac{1}{2}(X_1X_2 + X_2X_1)$, and similarly for y_k for all $k \geq 3$. It follows that $A(n)$ may be presented as

$$\frac{\mathbb{k}\langle X_1, \dots, X_n \rangle}{\langle X_1^2 + X_2^2, B_{ij}(X_1X_2 + X_2X_1) - (X_iX_j + \mu_{ij}X_jX_i) : 1 \leq i \neq j \leq n \rangle}.$$

This algebra is an iterated Ore extension of $\mathbb{k}\langle X_1, X_2 \rangle / \langle X_1^2 + X_2^2 \rangle$. For $k \geq 3$, the automorphisms σ_k used to adjoin X_k are given by $\sigma_k(X_i) = -\mu_{ki}X_i$, and the corresponding left σ_k -derivations are given by $\delta_k(X_i) = B_{ki}(X_1X_2 + X_2X_1)$. To see that σ_k is well defined, we use the hypothesis that $\mu_{k1}^2 = 1 = \mu_{k2}^2$ to obtain $\sigma_k(X_1^2 + X_2^2) = 0$, and we use condition \star to establish that $\sigma_k(B_{ij}(X_1X_2 + X_2X_1) - (X_iX_j + \mu_{ij}X_jX_i)) = 0$. Turning to δ_k , we calculate that, for any $i \neq j$, we have

$$\delta_k(X_iX_j + \mu_{ij}X_jX_i) = (B_{ki}(1 - \mu_{kj}\mu_{ij})X_j + B_{kj}(\mu_{ij} - \mu_{ki})X_i)(X_1X_2 + X_2X_1).$$

Condition \star implies that if $B_{ki} \neq 0$, then $\mu_{kj} = \mu_{ji} = \mu_{ij}^{-1}$, and if $B_{kj} \neq 0$, then $\mu_{ki} = \mu_{ij}$. Thus $\delta_k(X_i X_j + \mu_{ij} X_j X_i) = 0$ for all $i \neq j$. Similarly, $\delta_k(X_1^2 + X_2^2) = 0$. Hence δ_k is well defined. Since $\mathbb{k}\langle X_1, X_2 \rangle / \langle X_1^2 + X_2^2 \rangle$ is Auslander regular, and, by [15], Ore extensions preserve this property, $A(n)$ is also Auslander regular.

Case II: $B_{11} \neq 0$. By rescaling the x_i if necessary, we may assume that $B_{11} = 2$ while maintaining $B_{12} = 1$. Given that $A(n)$ is quadratic and given the definition of M_1, \dots, M_n , the defining relations of $A(n)$ are of the form: $2X_1^2 = 4y_1$, $2X_k^2 \in \mathbb{k}y_1 + \mathbb{k}^\times y_k$, $X_i X_j + \mu_{ij} X_j X_i = 2B_{ij}y_1$, for all $k \geq 2$, and $1 \leq i \neq j \leq n$. It follows that $A(n)$ may be presented as

$$\frac{\mathbb{k}\langle X_1, \dots, X_n \rangle}{\langle X_1 X_2 + X_2 X_1 - X_1^2, B_{ij}(X_1 X_2 + X_2 X_1) - (X_i X_j + \mu_{ij} X_j X_i) : 1 \leq i \neq j \leq n \rangle}.$$

Using the above formulae for the maps σ_k and δ_k , this algebra is an iterated Ore extension of

$$\frac{\mathbb{k}\langle X_1, X_2 \rangle}{\langle X_1 X_2 + X_2 X_1 - X_1^2 \rangle},$$

which is Auslander regular by [1]. In particular, an argument similar to that in Case I shows that the σ_k and δ_k are well defined (but, in computing $\delta_k(X_1 X_2 + X_2 X_1 - X_1^2)$, one should note that condition \star implies that if $B_{ki} \neq 0$, where $i = 1, 2$, then $\mu_{kj} = \mu_{ji}$, which equals 1 if $j \in \{1, 2\}$). Again, as in Case I, it follows from [15] that $A(n)$ is Auslander regular.

We conclude that, in all cases, $A(n)$ is a quadratic, noetherian, Auslander-regular domain. \square

Theorem 5.5 and Remark 3.2 imply that if B is an invertible diagonal matrix, then any corresponding skew Clifford algebra of full dimension is a quotient of a quadratic Auslander-regular algebra.

6. OTHER GENERALIZATIONS

In this section we compare skew Clifford algebras to other generalizations of classical Clifford algebras found in the literature.

There are at least three different constructions called ‘‘generalized Clifford algebras,’’ all of which extend the classical notion by allowing defining relations of degree greater than two. N. Roby’s generalization of Clifford algebras in [20] was subsequently extended by C. Pappacena in [19] in his study of matrix pencils. Pappacena shows that when the defining relations are degree three or higher, these generalized Clifford algebras have infinite dimension. He also proves that when the defining relations are quadratic, and $\text{char}(\mathbb{k}) \neq 2$, then these algebras are isomorphic to classical Clifford algebras.

C. Ko in [13] approaches the problem of generalizing Clifford algebras from a different direction. Given a vector space V with quadratic form Q , Ko looks for algebras A that share the following property of classical Clifford algebras.

Condition 1. *Every isometry of V with respect to the quadratic form Q can be extended to an automorphism of A , and conversely, every automorphism of A mapping V to itself induces an isometry on V with respect to Q .*

Koç describes a family of algebras that satisfy this condition and it contains, as the quadratic members, the classical Clifford algebras; the other algebras in Koç's family are not quadratic.

Morris, in [17], defines generalized Clifford algebras that are presented with generators x_1, \dots, x_n and relations $x_i x_j = \omega x_j x_i$ and $x_i^N = 1$, where ω is a primitive N^{th} root of unity. These generalized Clifford algebras, like those of Pappacena and Koç, coincide with our skew Clifford algebras only if the algebra is a classical Clifford algebra.

Skew Clifford algebras also bear some resemblance to the quantum Clifford algebras given in [10] (see also [2]). Quantum Clifford algebras are presented with $2n$ generators, $x_1, \dots, x_n, x_1^*, \dots, x_n^*$, with defining relations

$$\begin{aligned} x_i x_j &= -q^{-1} x_j x_i, & x_i^* x_j^* &= -q x_j^* x_i^* \quad (i > j), \\ x_i x_i &= 0 = x_i^* x_i^*, & x_i x_j^* &= -q^{-1} x_j^* x_i \quad (i \neq j), \\ x_i x_i^* + x_i^* x_i &= (q^{-2} - 1) \sum_{i < j} x_j x_j^* + 1. \end{aligned}$$

A quantum Clifford algebra is a skew Clifford algebra if $q = \pm 1$. The case $q = 1$ yields a classical Clifford algebra.

Closer in spirit to our skew Clifford algebras is the recent work of Chen and Kang in [9], which develops a generalized Clifford theory on group-graded vector spaces using bicharacters. Their construction is quite general, and includes classical Clifford algebras, Weyl algebras, and polynomial rings.

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