

THE DEFINING RELATIONS OF QUANTUM $n \times n$ MATRICES

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ABSTRACT. Let $\mathcal{O}_q(M_n)$ denote the coordinate ring of quantum $n \times n$ matrices. We show there exists a subvariety \mathcal{P}_n of $\mathbb{P}(M_n)$ and an automorphism σ_n of \mathcal{P}_n such that $\mathcal{O}_q(M_n)$ determines, and is determined by, the geometric data $\{\mathcal{P}_n, \sigma_n\}$. The linear span of the defining relations of $\mathcal{O}_q(M_n)$ is the set of all those elements of $M_n^* \otimes M_n$ that vanish on the graph of σ_n . Moreover, if $q^2 \neq 1$, the variety \mathcal{P}_n is independent of q . Our main result is that there are two natural descriptions of \mathcal{P}_n . Firstly, if $q \neq 0$, there is a natural bijection between \mathcal{P}_n and the point modules over $\mathcal{O}_q(M_n)$, and the automorphism σ_n is the shift functor on point modules. Secondly, since $\mathcal{O}_q(M_n)$ is a flat deformation of $\mathcal{O}_1(M_n)$, the polynomial ring $\mathcal{O}(M_n)$, there is a homogenous Poisson bracket on $\mathcal{O}(M_n)$, and an associated Poisson structure on $\mathbb{P}(M_n)$. In this context, if $q^2 \neq 1$, the variety \mathcal{P}_n consists of those points of $\mathbb{P}(M_n)$ which are the zero-dimensional symplectic leaves with respect to this Poisson structure.

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Fix an algebraically closed field k , with $\text{char}(k) \neq 2$. Let M_n denote the ring of $n \times n$ matrices over k , and, for $q \neq 0$, let $\mathcal{O}_q(M_n)$ denote the coordinate ring of quantum $n \times n$ matrices over k ([FRT]).

This paper gives a geometric description of the defining relations of $\mathcal{O}_q(M_n)$. The case $n = 2$ is treated in [V1] (see example 2.1). In 2.5, we show that the defining relations of $\mathcal{O}_q(M_n)$ determine, and are determined by, a subvariety \mathcal{P}_n of $\mathbb{P}(M_n)$ together with an automorphism σ_n of \mathcal{P}_n . In particular, the linear span of the defining relations of $\mathcal{O}_q(M_n)$ is the set of all those elements in $M_n^* \otimes M_n^*$ that vanish on the graph of σ_n . The results were derived by using the non-commutative algebraic geometry developed by Artin, Tate and Van den Bergh in [ATV1] (see §1.3).

For $q^2 \neq 1$, the variety \mathcal{P}_n has two natural descriptions – one via symplectic leaves and the other via point modules. Viewing $\mathcal{O}_q(M_n)$ as a flat deformation of the polynomial ring $\mathcal{O}_1(M_n)$ (which is the coordinate ring $\mathcal{O}(M_n)$ of $n \times n$ matrices), and taking the limit as $q \rightarrow 1$ (see §1.2), we find that $\mathcal{O}(M_n)$ carries the structure of a Poisson algebra with respect to a homogeneous Poisson bracket of degree zero ([D]). This induces a Poisson structure on $\mathbb{P}(M_n)$, with respect to which \mathcal{P}_n consists of those points of $\mathbb{P}(M_n)$ which are the zero-dimensional symplectic leaves (§3). On the other hand, in §2, we associate to each point of \mathcal{P}_n a module over $\mathcal{O}_q(M_n)$. A *point module* over $\mathcal{O}_q(M_n)$ is a cyclic graded $\mathcal{O}_q(M_n)$ -module with Hilbert series $(1 - x)^{-1}$ ([ATV1]). Let M be a point module over $\mathcal{O}_q(M_n)$. The degree one component of the kernel of the surjective homomorphism $\mathcal{O}_q(M_n) \rightarrow M$ is of codimension one in $\mathcal{O}_q(M_n)_1 (= M_n^*)$, and so the homomorphism determines a point of $\mathbb{P}(M_n)$. This point belongs to \mathcal{P}_n , and the results of §2 show that for all $q \neq 0$ there exists a natural bijection between \mathcal{P}_n and the isomorphism classes of point modules over $\mathcal{O}_q(M_n)$. This natural bijection shows that \mathcal{P}_n represents the functor “point modules with values in ...” which is described in [ATV1, 3.9]. Here the automorphism σ_n is the *shift functor* on point modules ([A], [ATV1]). Combining these results, for $q^2 \neq 1$, gives a natural bijection between the zero-dimensional symplectic leaves and the isomorphism classes of point modules over $\mathcal{O}_q(M_n)$. This shows that the Poisson structure on the polynomial ring $\mathcal{O}(M_n)$ yields information about the module theory of the “generic” deformation $\mathcal{O}_q(M_n)$.

Section 4 considers the R -matrix construction of an algebra $A(R)$, which yields $\mathcal{O}_q(M_n)$ for a certain matrix R ([FRT]). In 4.2, we show that if $(x, y) \in \mathbb{P}(M_n) \times \mathbb{P}(M_n)$, then (x, y) belongs to the zero locus of the defining relations of $A(R)$ if and only if $x \otimes y$ commutes with τR , where τ is the map: $\tau(u \otimes v) = v \otimes u$ for all $u, v \in k^n$.

The author’s thesis [V2] will consider more general (flat) deformations of the polynomial ring and will show that, under certain conditions, the variety parametrizing the point modules over the “generic” deformation is contained in the variety consisting of the zero-dimensional symplectic leaves.

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1. PRELIMINARIES

Fix throughout an indeterminate t and an integer $n \geq 2$. Let A be the $k[t, t^{-1}]$ -algebra on n^2 generators $\{x_{ij} : 1 \leq i, j \leq n\}$, with defining relations determined by the requirement that whenever $r < s$ and $l < m$ there exists a $k[t, t^{-1}]$ -algebra isomorphism:

$$k[t, t^{-1}] \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\sim} k[t, t^{-1}] \begin{bmatrix} x_{rl} & x_{rm} \\ x_{sl} & x_{sm} \end{bmatrix}, \quad (*)$$

where the $k[t, t^{-1}]$ -algebra $k[t, t^{-1}][a, b, c, d] = k[t, t^{-1}] \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has the six defining relations:

$$\begin{aligned} ab &= tba, & bd &= tdb, & bc &= cb, \\ cd &= tdc, & ac &= tca, & ad - da &= (t - t^{-1})bc. \end{aligned}$$

The algebra A is the coordinate ring of quantum $n \times n$ matrices over the ring $k[t, t^{-1}]$, and for nonzero $q \in k$ we have $A/A(t - q) = \mathcal{O}_q(M_n)$ ([FRT]). We write $\mathcal{O}(M_n)$ for $\mathcal{O}_1(M_n)$, the polynomial ring on n^2 variables.

1.1. The Maps Φ and Ψ .

Let $\{x_{ij} : 1 \leq i, j \leq n\}$ denote a basis for M_n^* . For each $r < s$ and $l < m$, we define maps

$$\Phi_{sm}^{rl} : M_n^* \rightarrow M_2^* \quad \text{and} \quad \Psi_{sm}^{rl} : M_2^* \rightarrow M_n^*,$$

by

$$\Phi_{sm}^{rl} \begin{bmatrix} x_{rl} & x_{rm} \\ x_{sl} & x_{sm} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \Phi_{sm}^{rl}(x_{ij}) = 0 \text{ for all } i \notin \{r, s\} \text{ or } j \notin \{l, m\}$$

and

$$\Psi_{sm}^{rl} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x_{rl} & x_{rm} \\ x_{sl} & x_{sm} \end{bmatrix}.$$

These maps induce k -algebra homomorphisms

$$\Phi_{sm}^{rl} : T(M_n^*) \rightarrow T(M_2^*) \quad \text{and} \quad \Psi_{sm}^{rl} : T(M_2^*) \rightarrow T(M_n^*),$$

where $T(M_n^*)$ denotes the tensor algebra on M_n^* . Let R_2 denote the subspace of $M_2^* \otimes M_2^*$ generated by the defining relations of $\mathcal{O}_q(M_2)$, and define

$$R_n = \sum_{\substack{1 \leq r < s \leq n, \\ 1 \leq l < m \leq n}} \Psi_{sm}^{rl}(R_2).$$

Comparing this with (*), we have $\mathcal{O}_q(M_n) \cong T(M_n^*)/\langle R_n \rangle$. Notice that $\Phi_{sm}^{rl} \circ \Psi_{sm}^{rl}|_{M_2^*}$ is the identity on M_2^* , and hence on $T(M_2^*)$, and that $\Phi_{sm}^{rl} \circ \Psi_{s'm'}^{r'l'} = 0$ if $(r, s, l, m) \neq (r', s', l', m')$. Therefore, $\Psi_{sm}^{rl}(R_2) \subset R_n$ by definition of R_n , and

$$\Phi_{sm}^{rl}(R_n) = \sum_{\substack{1 \leq r' < s' \leq n, \\ 1 \leq l' < m' \leq n}} \Phi_{sm}^{rl} \circ \Psi_{s'm'}^{r'l'}(R_2) = \Phi_{sm}^{rl} \circ \Psi_{sm}^{rl}(R_2) = R_2.$$

It follows that, for each $r < s$ and $l < m$, $\Phi_{sm}^{rl} : T(M_n^*) \rightarrow T(M_2^*)$ and $\Psi_{sm}^{rl} : T(M_2^*) \rightarrow T(M_n^*)$ induce k -algebra homomorphisms

$$\phi_{sm}^{rl} : \mathcal{O}_q(M_n) \twoheadrightarrow \mathcal{O}_q(M_2) \quad \text{and} \quad \psi_{sm}^{rl} : \mathcal{O}_q(M_2) \hookrightarrow \mathcal{O}_q(M_n).$$

The map ψ_{sm}^{rl} is a specialization of the map (*).

The maps ϕ_{sm}^{rl} induce corresponding comorphisms

$$f_{sm}^{rl} : \mathbb{P}(M_2) \hookrightarrow \mathbb{P}(M_n)$$

which embed $\mathbb{P}(M_2)$ as a linear subspace of $\mathbb{P}(M_n)$. The maps ψ_{sm}^{rl} induce corresponding comorphisms

$$\mathbb{P}(M_n) \setminus \mathcal{V}(x_{rl}, x_{rm}, x_{sl}, x_{sm}) \rightarrow \mathbb{P}(M_2).$$

Since ψ_{sm}^{rl} and ϕ_{sm}^{rl} are Poisson algebra homomorphisms, their corresponding comorphisms send symplectic leaves to symplectic leaves.

For $d \leq n$, let $D_d \subset M_d$ denote the set of diagonal $d \times d$ matrices. So $\mathcal{O}(D_d) \cong \mathcal{O}(k^d)$. Let $I = (i_1, \dots, i_d)$ and $J = (j_1, \dots, j_d)$ where $1 \leq i_1 < i_2 < \dots < i_d \leq n$ and $1 \leq j_1 < j_2 < \dots < j_d \leq n$. Let $\theta_{IJ} : D_d \hookrightarrow M_n$ denote the linear map sending $\text{diag}(\alpha_1, \dots, \alpha_d)$ to the matrix with α_k in the $i_k j_k$ position for each $k \in \{1, \dots, d\}$, and zeros elsewhere. For each I and J , the linear dual of θ_{IJ} induces a k -algebra homomorphism $T(M_n^*) \rightarrow T(D_d^*)$, which induces a k -algebra homomorphism $\mathcal{O}_q(M_n) \rightarrow \mathcal{O}(D_d)$.

The interplay between all these maps will be used in §2.

1.2. The Poisson Bracket on $\mathcal{O}(M_n)$.

Since A is a free $k[t, t^{-1}]$ -module, we may view $\mathcal{O}_q(M_n)$ as a (flat) deformation of $\mathcal{O}(M_n)$ via the algebra A . Since $\mathcal{O}_q(M_n)$ has a Poincaré-Birkhoff-Witt basis over k , the element $h := t - 1$ is a central, regular element of A . If $f, g \in \mathcal{O}(M_n)$ then $\tilde{f}\tilde{g} - \tilde{g}\tilde{f} \in Ah$ where \tilde{f}, \tilde{g} are preimages of f, g in A . With this notation, we define a Poisson bracket $\{ , \}$ on $\mathcal{O}(M_n)$ by:

$$\{f, g\} = h^{-1}(\tilde{f}\tilde{g} - \tilde{g}\tilde{f}) \quad \text{modulo } Ah$$

(see, for example, [D]), which is independent of the choice of preimages of f and g .

This bracket extends to $\mathcal{O}(M_n)[z^{-1}]$ for any homogeneous element $z \in \mathcal{O}(M_n)$, and since the bracket is homogeneous of degree zero, it restricts to the degree zero part $(\mathcal{O}(M_n)[z^{-1}])_0$. In this way, the bracket induces a Poisson structure on the projective space $\mathbb{P}(M_n)$. By standard results, the maximal connected components of $\mathbb{P}(M_n)$ on which the Poisson structure is nondegenerate are symplectic manifolds, and $\mathbb{P}(M_n)$ decomposes as the disjoint union of these symplectic manifolds, or *symplectic leaves* ([K]). The zero-dimensional symplectic leaves in $\mathbb{P}(M_n)$ are given by the vanishing of the Poisson bracket on $(\mathcal{O}(M_n)[z^{-1}])_0$ for all homogeneous $z \in \mathcal{O}(M_n)$. We introduce the term *symplectic points* to refer to the zero-dimensional symplectic leaves.

1.3. Non-commutative Algebraic Geometry.

We view M_n^* acting as linear forms on $\mathbb{P}(M_n)$, and $M_n^* \otimes M_n^*$ as bilinear forms on $\mathbb{P}(M_n) \times \mathbb{P}(M_n)$. Let

$$\Gamma_n := \{(x, y) \in \mathbb{P}(M_n) \times \mathbb{P}(M_n) : f(x, y) = 0 \text{ for all } f \in R_n\}.$$

The main results (2.3, 2.5) of §2 show that, for all $q \neq 0$, there is a variety $\mathcal{P}_n \subset \mathbb{P}(M_n)$ and an automorphism σ_n of \mathcal{P}_n such that Γ_n is the graph of σ_n . In 2.8, we show that for all $q \neq 0$ there is a natural bijection between \mathcal{P}_n and the point modules over $\mathcal{O}_q(M_n)$. The main result of the paper, proved in §3, is that if $q^2 \neq 1$, then the variety \mathcal{P}_n consists of those points of $\mathbb{P}(M_n)$ which are the symplectic points for the Poisson structure on $\mathbb{P}(M_n)$.

2. THE POINT VARIETY

We retain the notation of §1. In this section, we show there exists a subvariety \mathcal{P}_n of $\mathbb{P}(M_n)$ (called the point variety of $\mathcal{O}_q(M_n)$) and an automorphism σ_n of \mathcal{P}_n with graph $\Gamma_n \subset \mathbb{P}(M_n) \times \mathbb{P}(M_n)$ such that, for all $q \neq 0$,

$$\mathcal{O}_q(M_n) \cong \frac{T(M_n^*)}{\langle f \in M_n^* \otimes M_n^* : f(\Gamma_n) = 0 \rangle}.$$

For $q^2 \neq 1$, we show \mathcal{P}_n is independent of q . For $q = 1$, $\mathcal{O}_1(M_n)$ is the polynomial ring $\mathcal{O}(M_n)$, and since the defining relations are the skew-symmetric tensors, Γ_n is the diagonal, so $\mathcal{P}_n = \mathbb{P}(M_n)$ and σ_n is the identity map. Thus, loosely speaking, the geometric data $\{\mathcal{P}_n, \sigma_n\}$ “encodes” the non-commutativity of $\mathcal{O}_q(M_n)$. The case $q = -1$ is exceptional but its proof follows from that for $q^2 \neq 1$ (2.3, 2.5) – we discuss this case in remark 2.6.

Example 2.1. – the $n = 2$ case.

For $q^2 \neq 1$ and $n = 2$, it is shown in [V1] that $R_2 = \{f \in M_2^* \otimes M_2^* : f(\Gamma_2) = 0\}$, and that Γ_2 is the graph of an automorphism σ_2 of a subvariety \mathcal{P}_2 of $\mathbb{P}(M_2)$, where $\mathcal{P}_2 = Q \cup L$, $Q = \mathcal{V}(ad - bc)$ (the singular matrices in $\mathbb{P}(M_2)$), and $L = \mathcal{V}(b, c)$ (the diagonal matrices). On L , σ_2 is the identity map, and on Q

$$\sigma_2 \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} q^{-1}a & b \\ c & qd \end{bmatrix}.$$

The geometric data $\{L, \sigma_2|_L\}$ corresponds to the quotient of $\mathcal{O}_q(M_2)$ by the ideal $\langle b, c \rangle$ – the factor ring is isomorphic to the polynomial ring $\mathcal{O}(k^2)$ on two variables, which is the coordinate ring of the diagonal matrices.

The following remark is used in the proof of 2.3.

Remark 2.2. Consider the coordinate ring of quantum affine n -space, $\mathcal{O}_q(k^n)$, given by the algebra $k[y_1, \dots, y_n]$ with defining relations $y_i y_j = q y_j y_i$ for all $1 \leq i < j \leq n$ where $q \in k$, $q \neq 0$ ([FRT]). Let $\{e_i\}_{i=1}^n$ denote the dual basis to $\{y_i\}_{i=1}^n$ in $\mathbb{P}(k^n)$. For $n \geq 3$, and $q \neq 1$, the collection of points in $\mathbb{P}(k^n) \times \mathbb{P}(k^n)$ on which the defining relations of $\mathcal{O}_q(k^n)$ vanish is given by

$$\{(p, \tau_n(p)) : p = p_i e_i + p_j e_j, \tau_n(p) = p_i e_i + q p_j e_j \text{ for all } 1 \leq i < j \leq n, (p_i, p_j) \in \mathbb{P}^1\}.$$

This variety is the graph of the automorphism $\tau_n : \mathcal{L} \rightarrow \mathcal{L}$ where $\mathcal{L} = \bigcup_{1 \leq i < j \leq n} \ell_{ij}$ and ℓ_{ij}

is the line in $\mathbb{P}(k^n)$ through e_i and e_j ; that is, \mathcal{L} consists of those points of $\mathbb{P}(k^n)$ having at most two nonzero coordinates with respect to the coordinate functions y_1, \dots, y_n . The linear span of the defining relations are the only elements of $(k^n)^* \otimes (k^n)^*$ that vanish on the graph of τ_n . Moreover, \mathcal{L} is in natural bijection with the point modules over $\mathcal{O}_q(k^n)$, and so we refer to \mathcal{L} as the point variety of $\mathcal{O}_q(k^n)$. We remark that $\mathcal{O}_q(k^n)$ may be viewed as a (flat) deformation of $\mathcal{O}_1(k^n)$, the polynomial ring $\mathcal{O}(k^n)$ on n variables, inducing a Poisson structure on $\mathbb{P}(k^n)$. In this case, the variety consisting of the symplectic points for this Poisson structure is the point variety \mathcal{L} .

Recall the maps Φ_{sm}^{rl} and Ψ_{sm}^{rl} defined in §1.1. In the following, the term *submatrix* refers to the $d \times d$ matrix formed from a matrix in M_n by deleting $n - d$ rows and $n - d$ columns.

Proposition 2.3. *For all $n \geq 2$ and for all $q \neq 0$, Γ_n is the graph of an automorphism σ_n of a variety $\mathcal{P}_n \subset \mathbb{P}(M_n)$. If $q^2 \neq 1$, the variety \mathcal{P}_n is independent of q and is the (nondisjoint) union of the following varieties:*

- (a) $\binom{n}{2}^2$ copies of $Q \cup L$, namely $f_{sm}^{rl}(\mathcal{P}_2)$ for all $r < s$ and $l < m$;
- (b) $\binom{n}{d}^2$ copies of \mathbb{P}^{d-1} for all $d = 1, \dots, n$, namely $\theta_{IJ}(D_d)$ for all $I = (i_1, \dots, i_d)$, $J = (j_1, \dots, j_d)$ where $1 \leq i_1 < i_2 < \dots < i_d \leq n$ and $1 \leq j_1 < j_2 < \dots < j_d \leq n$.

Proof. The first statement for $q^2 = 1$ is discussed in remark 2.6. Henceforth assume $q^2 \neq 1$.

Since the $n = 2$ case is proved in [V1], we will assume $n \geq 3$. Suppose $((a_{ij}), (b_{ij})) \in \Gamma_n$. Fixing $m \in \{1, \dots, n\}$ and comparing the relations in $M_n^* \otimes M_n^*$ of the form $x_{mi} \otimes x_{mj} - qx_{mj} \otimes x_{mi}$ for $1 \leq i < j \leq n$ with the relations for quantum affine n -space (given in remark 2.2), we find that at most two entries of the m 'th row of both (a_{ij}) and (b_{ij}) are nonzero, and that $a_{mi} = 0$ if and only if $b_{mi} = 0$ for all $i = 1, \dots, n$. A similar argument holds for the columns, and hence there are at most two nonzero entries in each row and each column of (a_{ij}) and (b_{ij}) , and $a_{ij} = 0$ if and only if $b_{ij} = 0$ for all $i, j = 1, \dots, n$.

Suppose $((a_{ij}), (b_{ij})) \in \Gamma_n$. For all $r < s$ and $l < m$, we have $(\Phi_{sm}^{rl})^{-1}(R_n) = R_2$. It follows that the elements of R_2 vanish on the pair

$$\left(\begin{pmatrix} a_{rl} & a_{rm} \\ a_{sl} & a_{sm} \end{pmatrix}, \begin{pmatrix} b_{rl} & b_{rm} \\ b_{sl} & b_{sm} \end{pmatrix} \right) \in \mathbb{P}(M_2) \times \mathbb{P}(M_2).$$

Hence, this pair of 2×2 matrices belongs to $\Gamma_2 \subset \mathbb{P}(M_2) \times \mathbb{P}(M_2)$. It follows that these matrices are of the form

$$\left(\begin{pmatrix} a_{rl} & 0 \\ 0 & a_{sm} \end{pmatrix}, \begin{pmatrix} a_{rl} & 0 \\ 0 & a_{sm} \end{pmatrix} \right), \quad \text{or} \quad \left(\begin{pmatrix} a_{rl} & a_{rm} \\ a_{sl} & a_{sm} \end{pmatrix}, \begin{pmatrix} q^{-1}a_{rl} & a_{rm} \\ a_{sl} & qa_{sm} \end{pmatrix} \right),$$

where the latter pair of submatrices are singular. This describes all the 2×2 submatrix-tuples of $((a_{ij}), (b_{ij})) \in \Gamma_n$.

By combining these results, it follows that $a_{11}a_{12}a_{23} = 0 = a_{11}a_{22}a_{23}$, and the ‘‘transpose’’ equation also holds; that is, $a_{11}a_{21}a_{32} = 0 = a_{11}a_{22}a_{32}$. Similarly for the other 2×3 , and 3×2 , submatrices of (a_{ij}) . It follows that either (a_{ij}) consists of a nonzero diagonal submatrix with all other entries equal to zero, or (a_{ij}) consists of a 2×2 nonzero singular submatrix \mathbf{a} such that all other entries are equal to zero. Analysis of the 2×2 submatrix-tuples shows that in the first case, $(b_{ij}) = (a_{ij})$, and in the second $b_{ij} = 0$ if and only if $a_{ij} = 0$ with the nonzero 2×2 submatrix of (b_{ij}) being given by $\sigma_2(\mathbf{a})$.

It follows that if $((a_{ij}), (b_{ij})) \in \Gamma_n$, then (b_{ij}) is uniquely determined by (a_{ij}) and vice versa, and that the projections $\Gamma_n \rightarrow \mathbb{P}(M_n)$, of Γ_n onto the first or second component, are equal. Therefore, denoting the images of these projections by \mathcal{P}_n , we have that Γ_n is the graph of an automorphism σ_n of \mathcal{P}_n . The fact that \mathcal{P}_n has the desired form follows from the description of the (a_{ij}) that occur. \blacksquare

Remark 2.4. The automorphism σ_n can be described explicitly from the proof of 2.3. For matrices in \mathcal{P}_n described in 2.3(b), σ_n acts as the identity map, whereas for those in 2.3(a), σ_n is given by the action of σ_2 on the nonzero singular 2×2 submatrix.

Theorem 2.5. For all $q \neq 0$,

$$\mathcal{O}_q(M_n) \cong \frac{T(M_n^*)}{\langle f \in M_n^* \otimes M_n^* : f(\Gamma_n) = 0 \rangle}.$$

Proof. The case $q^2 = 1$ is discussed in remark 2.6. Henceforth assume $q^2 \neq 1$.

It follows from the definition of Γ_n in §1.3 that

$$R_n \subseteq \{f \in M_n^* \otimes M_n^* : f(\Gamma_n) = 0\}.$$

Thus, we need only prove the reverse inclusion.

For a contradiction, suppose there exists $f \in (M_n^* \otimes M_n^*) \setminus R_n$ such that $f(\Gamma_n) = 0$. We may assume f is a linear combination of tensors of the form $x_{ij} \otimes x_{uv}$ where $i \leq u$ and $j \leq v$, or $i < u$ and $j > v$. Then there exist integers r, s, l and m with $r < s$ and $l < m$ such that $\Phi_{sm}^{rl}(f)$ is nonzero and is not an element of R_2 . However,

$$(\Phi_{sm}^{rl}(f))(\Gamma_2) = (f \circ (f_{sm}^{rl} \times f_{sm}^{rl}))(\Gamma_2),$$

which is zero since $(f_{sm}^{rl} \times f_{sm}^{rl})(\Gamma_2) \subset \Gamma_n$. Hence, $\Phi_{sm}^{rl}(f) \in R_2$ by [V1, 1.4], contradicting the earlier assertion. ■

Remark 2.6. As remarked above, if $q = 1$, then $\mathcal{P}_n = \mathbb{P}(M_n)$ and σ_n is the identity map, so 2.5 holds in this case. If $q = -1$ and $n = 2$, $\mathcal{O}_{-1}(M_2)$ is a twist (in the sense of [ATV2, §8]) of the polynomial ring $\mathcal{O}(M_2)$, and so $\mathcal{P}_2 = \mathbb{P}(M_2)$. Here, σ_2 is given by

$$\sigma_2 \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \left(\begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \right).$$

For $q = -1$ and $n \geq 3$, a similar proof to that of 2.3 shows that a matrix in \mathcal{P}_n either consists of a 2×2 nonzero (possibly nonsingular) submatrix with all other entries equal to zero, or it contains at most one nonzero entry per row and at most one nonzero entry per column. In the latter case, any arrangement of the nonzero entries is possible, since no longer do we have the restriction (as in 2.3) that a 2×2 submatrix is diagonal or singular. Moreover, for the first case, σ_n is given by the action of σ_2 on the 2×2 submatrix (c.f. the action of σ_n for $q^2 \neq 1$ on the matrices in 2.3(a)), and in the second case, σ_n acts as the identity map (c.f. the action of σ_n for $q^2 \neq 1$ on the matrices in 2.3(b)). A similar proof to that of 2.5 shows that 2.5 also holds if $q = -1$.

Definition 2.7. ([ATV1]) A point module over $\mathcal{O}_q(M_n)$ is a cyclic, graded $\mathcal{O}_q(M_n)$ -module with Hilbert series $(1 - x)^{-1}$.

Corollary 2.8. For all $q \neq 0$, the point modules over $\mathcal{O}_q(M_n)$ are parametrized, up to isomorphism, by the variety \mathcal{P}_n .

Proof. The result follows from [ATV1, 3.13] and proposition 2.3 since Γ_n , the set of zeros of the defining relations, is the graph of an automorphism of \mathcal{P}_n . ■

This correspondence may be described as follows. Given a vector space $M = \bigoplus_{i \geq 0} kv_i$, we may define an $\mathcal{O}_q(M_n)$ -module action on M by declaring $x_{ij} \cdot v_m = x_{ij}(\sigma_n^{-m}(p))v_{m+1}$, for all i, j, m , and some fixed $p \in \mathcal{P}_n$. Conversely, given a point module M over $\mathcal{O}_q(M_n)$, a basis for M exists such that the module action may be described in this way.

Moreover, in the category of graded modules over $\mathcal{O}_q(M_n)$ (modulo the torsion modules) the automorphism σ_n acts as the shift functor on point modules ([A], [ATV1]). That is, if $M(p) = \bigoplus_{i \geq 0} M_i$ represents a point module corresponding to the point $p \in \mathcal{P}_n$, then the point module $\bigoplus_{i \geq 1} M_i$ is isomorphic to the representative $M(\sigma_n^{-1}(p))$ of a point module corresponding to the point $\sigma_n^{-1}(p) \in \mathcal{P}_n$.

Henceforth, we refer to the variety \mathcal{P}_n as the *point variety* of $\mathcal{O}_q(M_n)$.

3. THE SYMPLECTIC POINTS

In this section, we show that the point variety \mathcal{P}_n , found in §2, consists of the symplectic points for the induced Poisson structure on $\mathbb{P}(M_n)$ described in §1.

Recall the definition of the Poisson bracket given in §1.1. On the generators x_{ij} of $\mathcal{O}(M_n)$ the bracket becomes:

$$\left. \begin{aligned} \{x_{im}, x_{jl}\} &= 0 \\ \{x_{il}, x_{jm}\} &= 2x_{im}x_{jl} \\ \{x_{il}, x_{im}\} &= x_{il}x_{im} \\ \{x_{il}, x_{jl}\} &= x_{il}x_{jl} \end{aligned} \right\} \quad \text{if } i < j \text{ and } l < m.$$

Example 3.1. – the $n = 2$ case.

Writing $\mathcal{O}_q(M_2) = k[a, b, c, d] = k \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have

$$\{b, c\} = 0, \quad \{a, d\} = 2bc, \quad \{a, b\} = ab, \quad \{a, c\} = ac, \quad \{b, d\} = bd, \quad \{c, d\} = cd.$$

Furthermore, the Poisson bracket extends to the ring of fractions $\mathcal{O}(M_n)[z^{-1}]$ and restricts to the degree zero part, $(\mathcal{O}(M_n)[z^{-1}])_0$, where z is a nonzero homogeneous element of $\mathcal{O}(M_n)$. The symplectic points in $\mathbb{P}(M_n)$ are the points which are the zeros for the brackets $\left\{ \frac{x}{z}, \frac{y}{z} \right\}$ on $(\mathcal{O}(M_n)[z^{-1}])_0$ for all homogeneous elements x, y, z of the same degree.

Proposition 3.2. *The variety in $\mathbb{P}(M_n)$ consisting of the symplectic points is the point variety \mathcal{P}_n of $\mathcal{O}_q(M_n)$ for $q^2 \neq 1$.*

Proof. Let \mathcal{P}'_n denote the variety in $\mathbb{P}(M_n)$ defined by the symplectic points; that is, the variety defined by the vanishing of all the brackets $\left\{ \frac{x}{z}, \frac{y}{z} \right\}$, where x, y, z are homogeneous and $\deg(x) = \deg(y) = \deg(z)$. However,

$$\left\{ \frac{x}{z}, \frac{y}{z} \right\} = z^{-3} (\{x, y\}z + \{y, z\}x + \{z, x\}y),$$

so \mathcal{P}'_n is the variety defined by the vanishing of all the polynomials:

$$\{x, y\}z + \{y, z\}x + \{z, x\}y.$$

Let $[x, y, z]$ denote this polynomial. Since it is invariant under cyclically permuting the variables, it suffices to find the zeros of at most $\binom{n^2}{3}$ polynomials.

For $n = 2$, \mathcal{P}'_2 is the zero locus of the four polynomials:

$$\begin{aligned} [a, b, c] &= 0, & [a, b, d] &= 2b(ad - bc), \\ [b, c, d] &= 0, & [a, c, d] &= 2c(ad - bc), \end{aligned} \tag{1}$$

giving $\mathcal{P}'_2 = Q \cup L$, which is \mathcal{P}_2 . It follows that, for all $n \geq 2$, the 2×2 submatrices of a matrix in \mathcal{P}'_n are either diagonal or singular.

Now, suppose $n \geq 3$ and $p \in \mathcal{P}'_n$. Focusing on the m 'th row and m 'th column of p in turn shows that the polynomials

$$[x_{mi}, x_{mj}, x_{ml}] = x_{mi}x_{mj}x_{ml} \quad \text{and} \quad [x_{im}, x_{jm}, x_{lm}] = x_{im}x_{jm}x_{lm} \tag{2}$$

vanish at p . This proves that p has at most two nonzero entries per row and at most two nonzero entries per column. Next, consider a 2×3 block of p . To ease notation, consider the first 2 rows and first 3 columns of p . Then the polynomials

$$\begin{aligned} [x_{11}, x_{12}, x_{23}] &= x_{11}x_{12}x_{23} + 2x_{13}(x_{11}x_{22} - x_{12}x_{21}), \\ [x_{11}, x_{22}, x_{23}] &= x_{11}x_{22}x_{23} + 2x_{12}(x_{12}x_{23} - x_{13}x_{22}) \end{aligned} \tag{3}$$

vanish at p . Combining (1), (2) and (3) implies that $x_{11}x_{12}x_{23}(p) = 0 = x_{11}x_{22}x_{23}(p)$. Similarly, for the other 2×3 submatrices of p . Moreover, the “transpose” polynomials $[x_{11}, x_{21}, x_{32}]$, $[x_{11}, x_{22}, x_{32}]$ to (3) also vanish at p , and similarly for the other 3×2 submatrices of p . It follows that $\mathcal{P}'_n \subset \mathcal{P}_n$.

To complete the proof, it suffices to show that the “diagonal” polynomials,

$$[x_{uu+m}, x_{vv+m}, x_{ww+m}],$$

all vanish on \mathcal{P}_n . Let $p \in \mathcal{P}_n$, and consider

$$[x_{11}, x_{22}, x_{33}] = 2(x_{11}x_{23}x_{32} - x_{22}x_{31}x_{13} + x_{33}x_{12}x_{21})$$

at p . Since p consists of either a 2×2 nonzero singular submatrix or a diagonal submatrix, it follows that $[x_{11}, x_{22}, x_{33}]$ vanishes at p . A similar argument holds for the other diagonal polynomials – proving they also vanish at p . Hence, $\mathcal{P}_n = \mathcal{P}'_n$. ■

Corollary 3.3. *If $q^2 \neq 1$, then the isomorphism classes of point modules over $\mathcal{O}_q(M_n)$ are in natural bijection with the symplectic points in $\mathbb{P}(M_n)$.*

Proof. Combine 2.8 and 3.2. ■

4. THE RELATIONSHIP BETWEEN Γ_n AND THE R -MATRIX

Let $R \in M_n \otimes M_n$. Following [FRT], we define below a quadratic algebra $A(R)$ which yields $\mathcal{O}_q(M_n)$ if

$$R = q \sum_{i=1}^n e_{ii} \otimes e_{ii} + \sum_{\substack{i,j=1 \\ i \neq j}}^n e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{\substack{i,j=1 \\ i > j}}^n e_{ij} \otimes e_{ji}, \quad (\ddagger)$$

where $\{e_{ij}\}$ is the usual basis of M_n . The algebra $A(R)$ is a quotient of $T(M_n^*)$, and we denote by $\Gamma(R)$ the zero locus in $\mathbb{P}(M_n) \times \mathbb{P}(M_n)$ of the quadratic relations defining $A(R)$. Theorem 4.2 gives a succinct description, in terms of R , of the pair of matrices (x, y) which belong to $\Gamma(R)$.

Let T denote $T(M_n^*)$, the tensor algebra on M_n^* . As in section 1, we fix a basis $\{x_{ij}\}$ for $T(M_n^*)_1 (= M_n^*)$, and define

$$\begin{aligned} X &= \sum_{i,j=1}^n e_{ij} \otimes x_{ij} && \in M_n \otimes T, \\ X_1 &= \sum_{i,j,m=1}^n e_{ij} \otimes e_{mm} \otimes x_{ij} && \in M_n \otimes M_n \otimes T \quad \text{and} \\ X_2 &= \sum_{i,j,m=1}^n e_{mm} \otimes e_{ij} \otimes x_{ij} && \in M_n \otimes M_n \otimes T. \end{aligned}$$

We consider R as an element of $M_n \otimes M_n \otimes T$ by identifying R with $R \otimes 1$. There is a natural k -algebra isomorphism

$$M_n \otimes M_n \otimes T \rightarrow M_{n^2} \otimes T \rightarrow M_{n^2}(T)$$

where $M_{n^2}(T)$ is the ring of $n^2 \times n^2$ matrices over T . Let $I(R)$ denote the ideal of T generated by the n^4 entries of the matrix

$$RX_1X_2 - X_2X_1R, \quad (*)$$

where the multiplication takes place in $M_{n^2}(T)$. Since these matrix entries are homogeneous elements of degree two, $I(R)$ is generated by its degree two component $I(R)_2 \subset M_n^* \otimes M_n^*$.

Definition 4.1. ([FRT]) We define

$$A(R) := \frac{T(M_n^*)}{I(R)}.$$

Theorem 4.2. Define $\tau: k^n \otimes k^n \rightarrow k^n \otimes k^n$ by $\tau(u \otimes v) = v \otimes u$, and consider τ as an element of $M_n \otimes M_n$. Let

$$\Gamma(R) := \{(x, y) \in \mathbb{P}(M_n) \times \mathbb{P}(M_n) : f(x, y) = 0 \text{ for all } f \in I(R)_2\}.$$

We have

$$\Gamma(R) = \{(x, y) \in \mathbb{P}(M_n) \times \mathbb{P}(M_n) : (\tau R)(x \otimes y) = (x \otimes y)(\tau R)\}.$$

Remark 4.3. This theorem was also proved independently by S. P. Smith.

Proof. By viewing the x_{ij} as homogeneous coordinates on $\mathbb{P}(M_n)$, it follows from (*) that $(x, y) \in \Gamma(R)$ if and only if

$$R(X_1|_x)(X_2|_y) - (X_2|_x)(X_1|_y)R = 0,$$

where $X_i|_x$, resp. $X_i|_y$, means evaluation at x , resp. y . Thus, $(x, y) \in \Gamma(R)$ if and only if

$$R(x \otimes 1)(1 \otimes y) = (1 \otimes x)(y \otimes 1)R$$

in M_{n^2} . Equivalently, (since the map $M_n \otimes M_n \rightarrow M_{n^2}$ is a ring homomorphism) we have $(x, y) \in \Gamma(R)$ if and only if

$$R(x \otimes y) = (y \otimes x)R,$$

where $x \otimes y, y \otimes x \in \mathbb{P}(M_n \otimes M_n)$ are the images of $(x, y), (y, x)$ respectively under the Segre embedding $\mathbb{P}(M_n) \times \mathbb{P}(M_n) \rightarrow \mathbb{P}(M_n \otimes M_n)$. However the map τ satisfies $y \otimes x = \tau(x \otimes y)\tau$ for all $(x, y) \in \mathbb{P}(M_n) \times \mathbb{P}(M_n)$. Thus, $(x, y) \in \Gamma(R)$ if and only if

$$R(x \otimes y) = \tau(x \otimes y)\tau R.$$

Applying τ to the left of both sides completes the proof. ■

Remark 4.4. Theorem 4.2 is reminiscent of an *enhancement* of τR , where τR satisfies the quantum Yang-Baxter equation (see [T, §2]); an enhancement of τR requires (amongst other things) a choice of matrix $y \in M_n$ such that $y \otimes y$ commutes with τR (and hence $y \otimes y$ also commutes with R , since $\tau(y \otimes y)\tau = y \otimes y$ for all $y \in M_n$). Write $y = (y_{ij})$ and suppose $y \otimes y$ commutes with τR . Then the assignment $x_{ij} \mapsto y_{ij}$ gives a one dimensional module over $A(R)$, since the image of the matrix (*) under this assignment is $R(y \otimes y) - (y \otimes y)R$, which is zero. Hence, an enhancement of τR involves a choice of one dimensional module over $A(R)$. Furthermore, any $y \in \mathbb{P}(M_n)$ such that $y \otimes y$ commutes with τR determines a point module $M(y)$ over $A(R)$ which is fixed by the shift functor and, conversely, if M is a point module fixed by the shift functor then it is isomorphic to some $M(y)$ where $y \otimes y$ commutes with τR . Every one dimensional $A(R)$ -module is a quotient of such a point module $M(y)$. This explains the geometric significance of the first condition given in [T, §2] for an enhancement of τR . In the case of $\mathcal{O}_q(M_n)$, by 2.4, every x given by 2.3(b) satisfies the first condition required for an enhancement of τR .

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