CONSTRUCTING CLIFFORD QUANTUM $\mathbb{P}^3$S WITH FINITELY MANY POINTS

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Abstract. We present an algebro-geometric technique for constructing regular Clifford algebras A of global dimension four with associated point scheme consisting of a prespecified finite number of points. In particular, if A has more than one point in its point scheme, then the number of points in the point scheme can be obtained from the number of intersection points of two planar cubic divisors; these cubic divisors correspond to regular Clifford subalgebras of A of global dimension three. If A has exactly a finite number, n, of distinct points in its point scheme, then $n \in \{1, 3, 4, 5, \ldots, 13, 14, 16, 18, 20\}$ and all these possibilities occur.

We also prove that if a regular Clifford algebra R of global dimension $d \geq 2$ has exactly a finite number, n, of distinct isomorphism classes of point modules, then n is odd if and only if R is an Ore extension of a regular Clifford subalgebra of R of global dimension $d-1$.

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Introduction

The classification of the Artin-Schelter regular algebras of global dimension four is an unsolved problem. Even the subproblem of classifying such algebras that are quadratic is unsolved. The standard methods to attack these problems entail geometric techniques that use certain modules in place of geometric objects, and, in many cases, the geometry so obtained depicts the algebraic structure of the algebra ([2, 3]). In this spirit, associated to such an algebra is the notion of *point scheme* and *line scheme* ([2, 13, 15]), and a regular algebra of global dimension four is often referred to as a *quantum* $\mathbb{P}^3$.

The current approach to these classification problems is to classify (or at least, better understand) the subclass of quadratic quantum $\mathbb{P}^3$s that have a finite point scheme and a one-dimensional line scheme. However, very few examples of such algebras are known, and even fewer are known that are infinite modules over their centers with their point scheme being the graph of an automorphism of finite order ([5, 8, 14]). This lack of examples is an obstacle to producing fresh conjectures in the subject.

One method for constructing quantum $\mathbb{P}^3$s that was successful in [16] involves first producing a regular Clifford algebra of global dimension four (herein called a *Clifford quantum* $\mathbb{P}^3$), and then deforming the defining relations of the algebra in some way. In [16], the Clifford algebra was an Ore extension of a regular Clifford algebra of global dimension three. The defining relations were modified in such a way that the non-commutative algebra was still an Ore extension of a regular algebra, and so it too was regular (but not a flat deformation of the Clifford algebra). The original Clifford algebra had only one point in its point scheme and a two-dimensional line scheme, while the deformed algebras also had one point in their point schemes but their line schemes had dimension one.

We wish to pursue a similar method for producing interesting new quantum $\mathbb{P}^3$s which have a finite point scheme and a one-dimensional line scheme. The first obstacle to overcome is to find a method for constructing Clifford quantum $\mathbb{P}^3$s with a prespecified number of distinct points. Hence, our main goal is to introduce a relatively simple algebro-geometric technique for accomplishing that task. In a sequel to this article, we will introduce methods for "deforming" such Clifford quantum $\mathbb{P}^3$s to produce quantum $\mathbb{P}^3$s of the desired type.

The standard way to count the number of points in the point scheme of a Clifford quantum $\mathbb{P}^3$ is described in [16] and entails associating a projective space of quadrics to the algebra and counting the number of quadrics of rank at most two in that projective quadric space.

This idea will be summarized in Section 1. Computationally, this method involves intersecting ten cubic surfaces in $\mathbb{P}^3$. In Section 2, we show that if the algebra has at least two points, then this method can be replaced with the intersection of two planar cubic divisors; the precise statement is given in Theorem 2.6. We also prove that usually there is a choice for the pair of cubic divisors to be used and that the cubic divisors correspond to regular Clifford subalgebras of global dimension three inside the original Clifford algebra (see Corollary 1.10 and Remark 1.11).

The cubic divisors allow us to prove in Section 3 that if a Clifford quantum $\mathbb{P}^3$ has exactly a finite number, n, of distinct points, then $n \notin \{2, 15, 17, 19\}$. Moreover, the cubic divisors provide a recipe for producing examples of Clifford quantum $\mathbb{P}^3$s with a prespecified value for n, and this is demonstrated in Section 5. In particular, all other values of $n \in \{1, \ldots, 20\}$ arise as the total number of distinct points of some Clifford quantum $\mathbb{P}^3$. In Section 4, in Theorem 4.6, we prove that if a regular Clifford algebra A of global dimension $d \geq 2$ has exactly an odd finite number of point modules, then A is an Ore extension of a regular Clifford algebra of global dimension $d - 1$, and conversely.

This article is laid out as follows. Section 1 gives the relevant definitions and summarizes results on Clifford algebras relevant to the results to be proved; the cubic divisors are introduced in that section. The following section uses the cubic divisors to count the number of points of a Clifford quantum $\mathbb{P}^3$ in Theorem 2.6. The main results of Section 3 concern the allowed values of n mentioned above, while the following section discusses the case that n is odd. In that section, in Corollary 4.8, we classify Clifford quantum $\mathbb{P}^2$s, up to isomorphism, since, if n is odd, Theorem 4.6 shows that a Clifford quantum $\mathbb{P}^3$ is an Ore extension of a Clifford quantum $\mathbb{P}^2$. The final section, Section 5, shows how to use the cubic divisors to produce examples of Clifford quantum $\mathbb{P}^3$s with a prespecified number of distinct points.

1. Construction of the Cubic Divisors

In this section, we first give definitions and state past results. We introduce certain cubic divisors associated with a Clifford quantum $\mathbb{P}^3$, and relate them to regular Clifford subalgebras. A choice of coordinates is also introduced for allowing computations with the cubic divisors later in the article.

1.1. **Preliminaries.**

Throughout the article, k denotes an algebraically closed field and $\mathrm{char}(k) \neq 2$. We use $\mathbb{N}$

to denote the nonnegative integers. If R is a ring or vector space, $R^\times$ denotes $R \setminus \{0\}$. The polynomial ring on d generators of degree one is denoted by S, so that $\operatorname{Proj} S = \mathbb{P}^{d-1}$. We use S_i for the vector space spanned by the homogeneous polynomials in S of degree i. (Occasionally, two polynomial rings with different values of d appear in the same proof; in this case, we denote one by S and the other by R to distinguish them.) All algebras appearing in the article are assumed to be $\mathbb{N}$-graded and locally finite. If A is such an algebra, then the shift $M[s]$ of a graded A-module M by an integer s is the graded A-module $M[s] = \bigoplus_i M_{s+i}$. If $j \in \mathbb{Z}$, the submodule $M_{\geq j}$ of a graded A-module M denotes the module $M_{\geq j} = \bigoplus_{i \geq j} M_i$.

Definition 1.1. [1] A connected, $\mathbb{N}$-graded, associative k-algebra A that is generated by degree-one elements is called *Artin-Schelter regular of dimension d* if A has finite global dimension d and polynomial growth, and if

$$\operatorname{Ext}^i_A(k, A) = \begin{cases} 0 & \text{if } i \neq d \\ k[s] & \text{if } i = d, \end{cases}$$

where $k[s]$ denotes the trivial, graded module k with its grading shifted by $s \in \mathbb{Z}$.

Such an algebra is viewed as being a non-commutative analogue of S. An alternative notion of regularity, called *Auslander-regularity*, is given in [10] and is used briefly in the sequel. An Auslander-regular algebra with polynomial growth is Artin-Schelter regular, but it is not known if the converse holds. Other well-behaved homological properties motivated by commutative algebra, such as the notion of an algebra satisfying the *Cohen-Macaulay property*, are discussed in [10]. These homological properties are satisfied by regular Clifford algebras, which are the focus of this article.

Definition 1.2. [9] Let $y_1, \ldots, y_d$ denote commuting indeterminates and let $Y = \sum_{m=1}^{d} Q_m y_m$, where the $Q_m \in M_d(k)$ are linearly independent symmetric matrices. The *Clifford algebra* associated to Y is the k-algebra on generators $x_1, \ldots, x_d$, $y_1, \ldots, y_d$ with defining relations $x_i x_j + x_j x_i = Y_{ij}$ for all i, j, and y_i central for all i. We define a grading on this algebra by declaring $\deg(x_i) = 1$ and $\deg(y_i) = 2$ for all i.

By [12], we may describe a *regular* Clifford algebra of global dimension d as the Koszul dual of a commutative quadratic algebra. To do so, form the quadratic commutative algebra $B = S/\langle q_1, \ldots, q_d \rangle$, where $q_1, \ldots, q_d \in S_2$. We write $V = S_1$ and let $T(V)$ denote the tensor algebra on V, and let U denote the subspace of skew-symmetric tensors in $T(V)_2$. Hence,

$$S = \frac{T(V)}{\langle U \rangle} \quad \text{and} \quad B = \frac{T(V)}{\langle U + kq_1 + \cdots + kq_d \rangle}.$$

Writing W for $(U + kq_1 + \cdots + kq_d)^\perp \subset V^* \otimes V^*$, we form the Koszul dual, A, of B by setting

$$A = \frac{T(V^*)}{\langle W \rangle}.$$

To each nonzero $q_i \in S_2$, one may associate the quadric $\mathcal{V}(q_i) \subset \mathbb{P}(V^*)$. Hence, to the algebra A, one may associate a linear system $\mathfrak{Q}$ of quadrics, where $\mathfrak{Q}$ is generated by the nontrivial quadrics in $\{\mathcal{V}(q_1), \ldots, \mathcal{V}(q_d)\}$. A point in the intersection, $\bigcap_{i=1}^{d} \mathcal{V}(q_i) \subset \mathbb{P}(V^*)$, of all the quadrics in $\mathfrak{Q}$ is called a *base point* of $\mathfrak{Q}$; if no such point exists, then $\mathfrak{Q}$ is said to be *base-point free.*

If $\dim_k(\mathfrak{Q}) = d$, then $\mathfrak{Q}$ determines d linearly independent symmetric matrices (up to conjugacy), and conversely; so $\mathfrak{Q}$ determines a Clifford algebra (possibly not regular nor given by A) and conversely.

Proposition 1.3.

(a) [9] *A Clifford algebra is quadratic and Auslander-regular and satisfies the Cohen-Macaulay property if and only if the associated d-dimensional quadric system $\mathfrak{Q}$ is base-point free.*

(b) [12] *If $\dim_k(\mathfrak{Q}) = d$ and if $\mathfrak{Q}$ is base-point free, then the Clifford algebra associated to $\mathfrak{Q}$ is the algebra A above, and all regular Clifford algebras are of this form. In the notation of Definition 1.2, the basis of V dual to $\{x_1, \ldots, x_d\}$ is the basis with respect to which the Q_m are written.*

(c) [4, Section 3] *With $\mathfrak{Q}$ as in (b), the associated Clifford algebra is the enveloping algebra of a Lie superalgebra and has Hilbert series $H(t) = 1/(1-t)^d$.* ■

As discussed in the introduction, the geometry associated to regular algebras involves certain modules replacing points, lines, etc. The substitute for a point is a *point module.*

Definition 1.4. [2] If A is a connected, $\mathbb{N}$-graded k-algebra which is generated by degree-one elements, then a *point module* over A is a graded cyclic module M over A that has Hilbert series $H_M(t) = 1/(1-t)$ such that M is generated by a homogeneous degree-zero element.

In [16], a formula was introduced for counting the number of isomorphism classes of point modules over a regular Clifford algebra A of global dimension d. If, as above, $\mathfrak{Q}$ denotes the base-point-free quadric system associated to A, and if r_i denotes the number of distinct quadrics in $\mathbb{P}(\mathfrak{Q})$ of rank i, then, by [16, Theorem 1.7], the number of isomorphism classes of point modules over A is given by $r_1+2r_2 \in \mathbb{N}\cup\{\infty\}$. Moreover, if the number of isomorphism classes of point modules is finite, then $r_1 \in \{0,1\}$. If $d = 2$ or 3, then $r_1 + 2r_2$ is infinite; if $d = 4$, then, by intersection theory in $\mathbb{P}(S_2) = \mathbb{P}^9$, $r_1 + 2r_2 \neq 0$. By [16, Proposition 1.5 and Lemma 1.6], elements of $\mathfrak{Q}$ of rank one (respectively, rank two) correspond to point modules M over A with the property that $M[1]_{\geq 0} \cong M$ (respectively, $M[2]_{\geq 0} \cong M$), and conversely.

In the sequel, we will refer to a regular Clifford algebra of global dimension d as a Clifford quantum $\mathbb{P}^{d-1}$. It is conventional to call a linear system $\mathfrak{q}$ of quadrics, such that $\mathbb{P}(\mathfrak{q}) \cong \mathbb{P}^2$, a *net* of quadrics, and a linear system $\mathfrak{Q}$ of quadrics, such that $\mathbb{P}(\mathfrak{Q}) \cong \mathbb{P}^3$, a *web* of quadrics.

1.2. **The Cubic Divisors.**

In this subsection, we prove that if $\mathfrak{q}$ is a net of quadrics in $\mathbb{P}^3$ and if Q is a quadric in $\mathbb{P}^3$ of rank at most two, such that $Q \notin \mathfrak{q}$, then Q determines two cubic divisors in $\mathbb{P}(\mathfrak{q})$.

Lemma 1.5. *Suppose $d \geq 3$. Let $\mathfrak{q}$ denote a system of quadrics in $\mathbb{P}^{d-1}$ such that $\mathbb{P}(\mathfrak{q}) \cong \mathbb{P}^{d-2}$ and let P denote a hyperplane in $\mathbb{P}^{d-1}$. If F is the subscheme of $\mathbb{P}(\mathfrak{q})$ consisting of those quadrics in $\mathbb{P}(\mathfrak{q})$ that meet P in a degenerate quadric, then* $\dim(F) \geq d-3$*; if $F \neq \mathbb{P}(\mathfrak{q})$, then* $\deg(F) = d-1$.

Proof. Let P denote a hyperplane in $\mathbb{P}^{d-1}$, let D denote the scheme of degenerate quadrics on P, and let X denote the scheme in $\mathbb{P}(S_2)$ of quadrics Q in $\mathbb{P}^{d-1}$ such that $Q \cap P \in D$. We will prove that $\dim(X \cap \mathbb{P}(\mathfrak{q})) \geq d-3$.

Write $P = \mathcal{V}(L)$, where $L \in S_1$, and, for any quadric Q in $\mathbb{P}^{d-1}$, write $Q = \mathcal{V}(q)$, where $q \in S_2$. Let $\tilde{q} \in S_2/S_1L$ denote the image of q in $S/\langle L\rangle$. The scheme $\mathcal{V}(\tilde{q})$ is the quadric $Q \cap P$ on the hyperplane P, and $\mathcal{V}(\tilde{q}) \in D$ if and only if $\operatorname{rank}(\tilde{q}) \leq d-2$; that is, the determinant of any symmetric $(d-1)\times(d-1)$ matrix, $\mathfrak{m}_q$, representing $\tilde{q}$ is zero.

With this notation, we have that

$$X = \{q \in \mathbb{P}(S_2) : \det(\mathfrak{m}_q) = 0\},$$

and so $\dim(X) = \dim(\mathbb{P}(S_2)) - 1$ and $\deg(X) = d-1$. Hence,

$$\dim(X \cap \mathbb{P}(\mathfrak{q})) \geq \dim(\mathbb{P}(\mathfrak{q})) - 1 = d-3.$$

Thus, if $X \cap \mathbb{P}(\mathfrak{q}) \neq \mathbb{P}(\mathfrak{q})$, then $\mathbb{P}(\mathfrak{q})$ lies in general position with respect to X, and so we may apply Bézout's Theorem to yield $\deg(X \cap \mathbb{P}(\mathfrak{q})) = \deg(X) = d-1$. ■

Lemma 1.6. *Let $\mathfrak{q}$ denote a net of quadrics in $\mathbb{P}^3$, and let Q denote a rank-two quadric in $\mathbb{P}^3 \setminus \mathfrak{q}$.*

(a) *For each plane $P \subset Q$, the subscheme F of $\mathbb{P}(\mathfrak{q})$ consisting of the quadrics in $\mathbb{P}(\mathfrak{q})$ that meet P in a degenerate conic has positive dimension; if $F \neq \mathbb{P}(\mathfrak{q})$, then* $\deg(F) = 3$.

(b) *There exists $Q' \in \mathfrak{q}$ such that $Q' \cap Q$ consists of four lines, counted with multiplicity.*

Proof. (a) Clearly (a) follows from Lemma 1.5.

(b) Let P_1 and P_2 denote the distinct planes of Q, and, for each $i = 1, 2$, let F_i denote the subscheme of $\mathbb{P}(\mathfrak{q})$ consisting of those quadrics in $\mathbb{P}(\mathfrak{q})$ that meet P_i in a degenerate conic. It follows from (a) that $\dim(F_1 \cap F_2) \geq 1 + 1 - 2 = 0$, so that $F_1 \cap F_2$ contains at least one quadric Q'. Since $Q' \in F_1 \cap F_2$, Q' intersects each of the P_i in a degenerate conic; thus $Q' \cap Q$ consists of four lines, counted with multiplicity. ■

Remark 1.7. Let A denote a Clifford quantum $\mathbb{P}^3$, and suppose A has at least two, but at most finitely many, distinct isomorphism classes of point modules. It follows that $\mathfrak{Q}$ contains a rank-two quadric Q, so that we may write $\mathfrak{Q} = kQ \oplus \mathfrak{q}$ for some net $\mathfrak{q}$ of quadrics in $\mathfrak{Q}$, such that $Q \notin \mathfrak{q}$. (Clearly, the choice of $\mathfrak{q}$ is not unique.) We will show in Subsection 1.3 that, in this setting, the two schemes, F_1 and F_2, given in Lemma 1.6(a) have dimension one, and so are planar cubic divisors determined by Q and $\mathfrak{q}$, and are the cubic divisors mentioned in the introduction. We will prove in Section 2 that $|F_1 \cap F_2|$ determines the number of isomorphism classes of point modules over A.

1.3. Coordinate Computations with the Cubic Divisors.

Let $\mathfrak{Q}$ denote a web of quadrics in $\mathbb{P}^3$ corresponding to a Clifford quantum $\mathbb{P}^3$. Suppose that $\mathfrak{Q}$ contains a rank-two quadric $Q = \mathcal{V}(a_1a_2)$, where $a_1, a_2 \in S_1$ are linearly independent. We may assume that $\mathfrak{Q}$ is given by linearly independent elements $q_1, \ldots, q_4 \in S_2$, where $q_1 = a_1a_2$ and $kq_2 \oplus kq_3 \oplus kq_4$ determines a net $\mathfrak{q}$ of quadrics in $\mathfrak{Q}$ such that $Q \notin \mathfrak{q}$. For each $i = 1, 2$, and for each $j = 1, \ldots, 4$, let $q_j^{(i)}$ denote the image of q_j in S_2/S_1a_i. Since $\mathfrak{Q}$ is base-point free, it follows that, for each $i = 1, 2$, the nonzero elements $q_2^{(i)}$, $q_3^{(i)}$ and $q_4^{(i)}$ determine a net $\mathfrak{q}_i$ of quadrics in $\mathcal{V}(a_i) = \mathbb{P}^2$ that is base-point free; that is, the net $\mathfrak{q}_i$ is the isomorphic image of the net $\mathfrak{q}$ determined by the map $S_2 \to S_2/S_1a_i$.

An arbitrary element Q' of $\mathbb{P}(\mathfrak{Q})$ is given by a linear combination, $\sum_{j=1}^{4} X_jq_j$, where the $X_j \in k$. In other words, Q' may be represented by $(X_1, \ldots, X_4) \in \mathbb{P}(\mathfrak{Q}) = \mathbb{P}^3$. On the other hand, if $Q' \neq Q$, then, for each $i = 1, 2$, the element corresponding to the image of Q' in

$S_2/S_1 a_i$ is given by $\sum_{j=2}^{4} X_j q_j^{(i)}$, and hence by $(X_2, X_3, X_4) \in \mathbb{P}(\mathfrak{q}_i) \cong \mathbb{P}(\mathfrak{q}) = \mathbb{P}^2$. It follows that, with this choice of coordinates, we may realize both $\mathbb{P}(\mathfrak{q}_1)$ and $\mathbb{P}(\mathfrak{q}_2)$ as $\mathbb{P}(\mathfrak{q})$ in $\mathbb{P}(\mathfrak{Q})$ via the projection map onto the last three coordinates, that is, via the projection map onto the same copy of $\mathbb{P}^2$ in $\mathbb{P}(\mathfrak{Q})$. The underlying reason these coordinate structures coincide is that $\mathfrak{Q}$ is base-point free and $q_1 = a_1 a_2$.

With $Q = \mathcal{V}(a_1 a_2)$, for each $i = 1, 2$, the scheme F_i, from Remark 1.7, is the family of quadrics in $\mathbb{P}(\mathfrak{q})$ that meet $\mathcal{V}(a_i)$ in a degenerate conic. Moreover, since $\mathfrak{Q}$ is base-point free, a quadric $Q' \in \mathbb{P}(\mathfrak{q})$ meets $\mathcal{V}(a_i)$ in a degenerate conic if and only if the image of Q' in $\mathbb{P}(\mathfrak{q}_i)$ has rank at most two. Hence, the scheme F_i parametrizes the quadrics in $\mathbb{P}(\mathfrak{q}_i)$ of rank at most two. This connection between F_i and $\mathbb{P}(\mathfrak{q}_i)$ will be used many times in the sequel.

Lemma 1.8. *With A, $\mathfrak{Q} \subset \mathbb{P}^3$, Q, $\mathfrak{q}$, $\mathfrak{q}_1$, $\mathfrak{q}_2$, F_1 and F_2 as above, with $\mathfrak{Q}$ base-point free, we have that F_i parametrizes the quadrics in $\mathbb{P}(\mathfrak{q}_i)$ of rank at most two and* $\dim(F_i) = 1$ *for* $i = 1, 2$*; that is, F_1 and F_2 are planar cubic divisors.*

Proof. Since $\mathfrak{Q}$ is base-point free, the result follows from the preceding discussion and Lemma 1.6(a). ■

1.4. **Subalgebras.**

In this subsection, we prove that a rank-two element of a linear system $\mathfrak{Q}$ of quadrics corresponding to a regular Clifford algebra A of global dimension d determines two Clifford subalgebras of A of global dimension $d - 1$.

Proposition 1.9. *Let A denote a regular Clifford algebra of global dimension d with corresponding quadric system $\mathfrak{Q}$ that is base-point free. Fix $\eta \in (A^!)_1 = V$ and let $\mathfrak{Q}'$ denote the quadric system obtained from $\mathfrak{Q}$ by intersecting each of the quadrics in $\mathfrak{Q}$ with the hyperplane $\mathcal{V}(\eta) \subset \mathbb{P}(V^*)$. If* $\dim_k(\mathfrak{Q}') = d - 1$, *then $\left(A^!/\langle\eta\rangle\right)^!$ is isomorphic to a subalgebra of A and is a regular Clifford algebra of global dimension $d - 1$.*

Proof. Under the map $S \to S/\langle\eta\rangle$, the image of the span of the quadratic forms that determine $\mathfrak{Q}$ is the span of the quadratic forms that determine $\mathfrak{Q}'$. Hence, $\mathfrak{Q}'$ is base-point free, since $\mathfrak{Q}$ is. This fact and the hypothesis that $\dim_k(\mathfrak{Q}') = d-1$ imply, by Proposition 1.3, that the Clifford algebra, A', corresponding to $\mathfrak{Q}'$ is quadratic and regular of global dimension $d-1$ with Hilbert series $1/(1-t)^{d-1}$. In particular, since A' is quadratic, and since the quadrics in $\mathfrak{Q}'$ lie in $\mathcal{V}(\eta)$, it follows, by [12], that $A' \cong (R/\langle Q\rangle)^!$, where $\operatorname{Proj} R = \mathcal{V}(\eta) = \mathbb{P}^{d-2}$ and $Q \subset R_2$ is the span of the quadratic forms in R_2 that determine $\mathfrak{Q}'$. However, $R/\langle Q\rangle \cong A^!/\langle \eta\rangle$, so $\big(A^!/\langle\eta\rangle\big)^! \cong A'$.

Let $y_1, \ldots, y_d$ denote commuting indeterminates, and let $y = \sum_{m=1}^{d} q_m y_m$ where $\{q_1, \ldots, q_d\} \subset S_2$ is linearly independent and determines $\mathfrak{Q}$. Since $\dim_k(\mathfrak{Q}') = d-1$, we may assume that $q_d \in S_1\eta$. We may choose a basis $\mathcal{B}$ for V that contains η and lists η last and which allows us to write $q_d \in k^\times\eta^2 + k\eta\epsilon$ where $\epsilon \in \operatorname{span}(\mathcal{B} \setminus \eta)$. We may represent y by a $d \times d$ symmetric matrix, Y, with respect to $\mathcal{B}$, that has the property that the $(d-1)\times(d-1)$ block, Y', of Y, formed by deleting the last column and the last row from Y, determines $\{q_m \text{ modulo } \langle\eta\rangle\}_{m=1}^{d}$. It follows that the entries of Y' span a subspace of $ky_1 + \cdots + ky_{d-1}$. By definition of $\mathfrak{Q}'$, we have that Y' determines $\mathfrak{Q}'$.

By using the dual basis $\{x_1, \ldots, x_d\} \subset V^*$ to $\mathcal{B}$, we have $x_d\eta \in k^\times$, and Definition 1.2 implies that Y determines the defining relations of A and that Y' determines the defining relations of A'. Since A' is quadratic with $(d-1)(d-2)/2$ relations (by Proposition 1.3), the entries of Y' span $ky_1 + \cdots + ky_{d-1}$ and $y_1, \ldots, y_{d-1} \in \sum_{i,j=1}^{d-1} k(x_ix_j + x_jx_i)$. The choice of $\mathcal{B}$ implies that $y_d \in kx_d^2 + ky_1 + \cdots + ky_{d-1}$, and so $x_ix_d + x_dx_i \in \sum_{j=1}^{d-1} ky_j + kx_d^2$ for all $i = 1, \ldots, d-1$. Hence, A has a spanning set $\{\mathcal{S}x_d^j\}_{j=0}^{\infty}$, where $\mathcal{S}$ is a spanning set for the subalgebra B of A generated by $\eta^\perp = kx_1 + \cdots + kx_{d-1}$. By Proposition 1.3, the Hilbert series of A is $1/(1-t)^d$, so the Hilbert series of B is at least $1/(1-t)^{d-1}$.

However, since the entries of Y' are entries of Y, it follows, from Definition 1.2, that the relations of A' are degree-two relations of B. Conceivably, B could have additional relations (even some with degree higher than two), and so we may only conclude that $A' \twoheadrightarrow B$ and that the Hilbert series of B is at most that of A', namely $1/(1-t)^{d-1}$. Combining this upper bound with our earlier lower bound, it follows that the Hilbert series of B equals $1/(1-t)^{d-1}$. Hence, $A' = B$, which completes the proof. ■

In Proposition 1.9, if, instead, $\dim(\mathfrak{Q}') = \dim(\mathfrak{Q})$, then $\mathfrak{Q}'$ does not determine a Clifford algebra, and, in this case, there exist examples where $\big(A^!/\langle\eta\rangle\big)^!$ is not a subalgebra of A. We thank the reviewer for drawing our attention to the necessity of the dimension hypothesis in Proposition 1.9.

Corollary 1.10. *Let A denote a regular Clifford algebra of global dimension d with corresponding quadric system $\mathfrak{Q}$ that is base-point free. If $\mathcal{V}(a_1a_2) \in \mathfrak{Q}$, where $a_1, a_2 \in S_1^\times$, then, for each $i = 1,\ 2$, $(A^!/A^!a_i)^!$ is a regular Clifford subalgebra of A of global dimension $d-1$.*

Proof. Fix $i \in \{1, 2\}$, and let $a_i = \eta$ in Proposition 1.9. Since $\mathfrak{Q}$ is base-point free and since $\mathcal{V}(a_1a_2) \in \mathfrak{Q}$, the dimension hypothesis of Proposition 1.9 is satisfied. The result follows. ■

Remark 1.11. Suppose $\mathfrak{Q}$ is base-point free and $d = 4$. By Lemma 1.8, for each $i = 1,\ 2$, the cubic divisor F_i parametrizes the quadrics of rank at most two in the net of quadrics corresponding to the regular Clifford subalgebra $(A^!/A^!a_i)^!$ in Corollary 1.10.

2. Counting Point Modules via Two Planar Cubic Divisors

In order to develop a theory that uses the number of intersection points of the two cubic divisors, F_1 and F_2 from Remark 1.7, for counting the number of point modules of a Clifford quantum $\mathbb{P}^3$, A, we must first relate the intersection points to rank-two elements of $\mathfrak{Q}$. We develop such a connection in this section and derive a formula relating $|F_1 \cap F_2|$ to the number of points modules over A.

Remark 2.1. Let F_1 and F_2 be as in Remark 1.7, so that each F_i is a planar cubic divisor (by Lemma 1.8), and $\deg(F_1 \cap F_2) = 9$, by Bézout's Theorem. Hence, if each of the quadrics in $F_1 \cap F_2$ were to have rank two, then, together with Q, there would be at least ten rank-2 quadrics in $\mathbb{P}(\mathfrak{q} \oplus kQ)$.

However, combining elements of $F_1 \cap F_2$ with Q could conceivably yield additional rank-two elements. Moreover, owing to the many choices for $\mathfrak{q}$, not all the quadrics Q' in $F_1 \cap F_2$ need have rank two, even though a linear combination of Q' and Q could have rank two, as the following result demonstrates.

Proposition 2.2. *Let Q and Q' denote distinct quadrics in $\mathbb{P}^3 = \operatorname{Proj} S$, where rank$(Q) = 2$ and rank$(Q') \geq 3$. Let ℓ denote the line on Q that is the intersection of the two planes of Q.*

(a) *Suppose $\ell \not\subset Q'$. If $kQ \oplus kQ'$ contains no rank-one quadrics, then there exists a rank-two quadric in $(kQ \oplus kQ') \setminus kQ$ if and only if $Q \cap Q'$ consists of four lines, counted with multiplicity.*

(b) *If* $\ell \not\subset Q'$*, then there exists a rank-one quadric in* $kQ \oplus kQ'$ *if and only if* $Q \cap Q'$ *consists of two double lines.*

(c) *Suppose* $\ell \subset Q'$*. If we write* $Q = \mathcal{V}(ab)$*, where* $a, b \in S_1$ *are linearly independent, then we may write* $Q' = \mathcal{V}(ac + bd)$*, where* $c, d \in S_1$*, with* $d = \alpha a + \beta b + \gamma c + \delta e$*, where* $e \in S_1$*,* $\alpha, \beta, \gamma, \delta \in k$*, and* a*,* b*,* c *and* e *are linearly independent. In this case, there is exactly one rank-two element in* $(kQ \oplus kQ') \setminus kQ$ *if and only if* $\delta = 0$ *and* $\gamma \neq 0$ *(and* $\beta \neq \alpha\gamma$ *since* $\text{rank}(Q') \geq 3$*), and the rank-two element is* $(a + \gamma b)(\beta b + \gamma c)$*; otherwise,* $(kQ \oplus kQ') \setminus kQ$ *contains no rank-two elements.*

Proof. The proof of (c) is straightforward and is left to the reader. The proof of (b) follows from Case 3 of the proof of (a) below. It remains to prove (a).

If there is a rank-two quadric $\bar{Q}$ in $(kQ \oplus kQ') \setminus kQ$, then $Q \cap Q' = Q \cap \bar{Q}$, and the latter consists of four lines counted with multiplicity (since $\ell \not\subset Q'$).

To consider the converse, let $Q = \mathcal{V}(ab)$, where $a, b \in S_1$ are linearly independent. Thus $\ell = \mathcal{V}(a, b)$. Write $\mathcal{V}(a) \cap Q' = \ell_a^1 \cup \ell_a^2$ and $\mathcal{V}(b) \cap Q' = \ell_b^1 \cup \ell_b^2$, where the ℓ_a^i and the ℓ_b^j are lines; since $\ell \not\subset Q'$, at least two of the potentially four lines are distinct, so there are at most three possible cases to consider.

Case 1: assume that all four lines are distinct. It follows that no three of the lines are coplanar. Let $c \in S_1$ be such that $\mathcal{V}(c)$ is the span of the lines ℓ_a^1 and ℓ_b^2, and let $d \in S_1$ be such that $\mathcal{V}(d)$ is the span of the lines ℓ_a^2 and ℓ_b^1. Since no three of the lines are coplanar, we may assume that a, b and c are linearly independent and $d \notin kc$. It follows that $\text{rank}(cd) = 2$ and $\mathcal{V}(cd) \cap Q = Q' \cap Q$, so $\mathcal{V}(cd) \in (kQ \oplus kQ') \setminus kQ$.

Case 2: assume that exactly three of the lines are distinct. Since $\ell \not\subset Q'$, we may assume that $\ell_b^1 = \ell_b^2$; that is, $\mathcal{V}(b) \cap Q'$ is a double line. It follows that $\text{rank}(Q') = 3$ and that $Q' = \mathcal{V}(bd + c^2)$ where $c, d \in S_1$ and b, c, d are linearly independent. Since exactly three of the lines are distinct, $d = \alpha a + \beta b + \gamma c$ for some $\alpha, \beta, \gamma \in k$ where $\gamma^2 \neq 4\beta$. This condition forces the quadric $\mathcal{V}(\beta b^2 + \gamma bc + c^2)$, which belongs to $(kQ \oplus kQ') \setminus kQ$, to have rank two.

Case 3: assume that exactly two of the lines are distinct. Since $\ell \not\subset Q'$, we need only consider $\ell_a^2 = \ell_a^1 \neq \ell_b^1 = \ell_b^2$. That is, $\mathcal{V}(a)$ intersects Q' in a double line, so that the quadratic form giving Q', when viewed modulo $S_1 a$, yields a rank-one element of $S_2/(S_1 a)$. Similarly for the intersection of $\mathcal{V}(b)$ with Q'. Let $c \in S_1$ be such that $\mathcal{V}(c)$ is the span of the lines ℓ_a^1 and ℓ_b^1. Hence, $Q \cap Q' = \mathcal{V}(a, c^2) \cup \mathcal{V}(b, c^2) = Q \cap V(c^2)$. Since these lines are divisors on Q and Q', it follows that $kQ \oplus kQ' = kQ + kc^2$, which contradicts our hypothesis in (a). ■

Lemma 2.3. *If Q and Q' are distinct quadrics in $\mathbb{P}^3$, then either $\mathbb{P}(kQ \oplus kQ')$ consists of quadrics of rank at most two or $\mathbb{P}(kQ \oplus kQ')$ contains at most three quadrics of rank at most two. Additionally, in the former case, if $Q = \mathcal{V}(ab)$, where $a, b \in S_1$ are linearly independent, then $Q' = \mathcal{V}(f)$ where $f \in S_1 a$ or $f \in S_1 b$ or $f \in k^\times a^2 + kab + k^\times b^2$.*

Proof. If $\mathbb{P}(kQ \oplus kQ')$ contains at most one quadric of rank at most two, then the result is proved. Hence, we may assume that $\mathbb{P}(kQ \oplus kQ')$ contains two distinct quadrics of rank at most two. If they both have rank one, say c^2 and d^2, where $c, d \in S_1$ are linearly independent, then $\mathbb{P}(kQ \oplus kQ')$ consists of quadrics of rank at most two. In this case, we may assume that $Q = \mathcal{V}(ab)$ where $a = c + \alpha d$ and $b = c - \alpha d$ for some $\alpha \in k$; if a and b are linearly independent, then $\alpha \neq 0$, so the result follows.

Hence, we may assume, after renaming, if necessary, that $\operatorname{rank}(Q) = 2$ and $\operatorname{rank}(Q') \leq 2$. Viewing Q and Q' as symmetric 4×4 matrices (in particular, as linear transformations from k^4 to k^4), if their images in k^4 intersect trivially, then $\mathbb{P}(kQ \oplus kQ')$ contains no additional rank-two quadrics and the result is proved. Instead, suppose that the images of Q and Q' intersect nontrivially in k^4. It follows that $\dim(\operatorname{im}(Q) + \operatorname{im}(Q')) \leq 3$. Since the image of every element of $kQ \oplus kQ'$ is contained in $\operatorname{im}(Q) + \operatorname{im}(Q')$, the symmetry of the matrices allows us to view them as symmetric 3×3 matrices (after a suitable basis change); that is, we may view $kQ \oplus kQ'$ as being contained in R_2, where $\mathbb{P}^2 = \operatorname{Proj} R$. Hence, it suffices to find the dimension and degree of the intersection in $\mathbb{P}(R_2)$ of $\mathbb{P}(kQ \oplus kQ')$ with the scheme of quadrics in $\mathbb{P}(R_2)$ of rank at most two. This intersection has dimension at least $1 + 4 - 5 = 0$, and so is nonempty. By Bézout's Theorem, its degree is the degree of the scheme of quadrics in $\mathbb{P}(R_2)$ of rank at most two, and that degree equals three. Thus, if the intersection has dimension zero, then it contains three elements counted with multiplicity. If the intersection has nonzero dimension, then $\mathbb{P}(kQ \oplus kQ') \cong \mathbb{P}^1$ is contained in the scheme of quadrics of rank at most two, and so $\mathbb{P}(kQ \oplus kQ')$ consists of quadrics of rank at most two. In this case, writing $Q = \mathcal{V}(ab)$, as in the statement of the lemma, it follows that $Q' = \mathcal{V}(f)$ where either $f \in S_1 a$ or $f \in S_1 b$ or $f \in k^\times a^2 + kab + k^\times b^2$. ■

In the generic case, $r_1 = 0$, so we may apply Proposition 2.2(a) and Lemma 2.3 to the setting of Remark 2.1 to count the number of rank-2 quadrics in $\mathbb{P}(kQ \oplus \mathfrak{q})$. The proof of Lemma 2.3 shows that, in the generic case, if $Q' \in F_1 \cap F_2$, then $\mathbb{P}(kQ \oplus kQ')$ contains exactly two distinct rank-two elements, with one being Q. By Remark 2.1, there are exactly nine distinct elements in $F_1 \cap F_2$, so it follows that $r_2 = 10$ in the generic setting. This agrees with [16, Theorem 2.1] and with the results in [17].

Corollary 2.4. *Suppose Q and Q' are quadrics in $\mathbb{P}^3$, where rank$(Q) = 2$ and rank$(Q') = 1$. If $\mathbb{P}(kQ \oplus kQ')$ contains a finite number of quadrics of rank at most two, then those quadrics are precisely Q and Q'.*

Proof. By Lemma 2.3, $\mathbb{P}(kQ \oplus kQ')$ contains at most three quadrics of rank at most two. As in the proof of Lemma 2.3, viewing Q and Q' as 4×4 symmetric matrices, their images in k^4 intersect trivially (otherwise $\mathbb{P}(kQ \oplus kQ')$ consists of quadrics of rank at most two). The result follows. ∎

Proposition 2.5. *Let $\mathfrak{q}$ denote a net of quadrics in $\mathbb{P}^3$, and let $Q = \mathcal{V}(ab) \notin \mathfrak{q}$, where $a, b \in S_1$ are linearly independent. Write F_1 and F_2 for the subschemes of $\mathbb{P}(\mathfrak{q})$ parametrizing the quadrics in $\mathbb{P}(\mathfrak{q})$ that meet the two distinct planes of Q in degenerate conics (as in the proof of Lemma 1.6).*

(a) *If $Q' \in F_1 \cap F_2$, then either $(kQ \oplus kQ') \setminus kQ$ contains an element of rank at most two or $Q' = \mathcal{V}(f)$ for some $f \in S_1 a + S_1 b$.*

(b) *Suppose that $kQ \oplus \mathfrak{q}$ is base-point free and $Q' \in \mathfrak{q}$. If $\alpha Q + \beta Q'$ has rank one, for some $\alpha,\ \beta \in k$, then Q' corresponds to a singular point on both cubic divisors F_1 and F_2.*

(c) *Suppose that $kQ \oplus \mathfrak{q}$ is base-point free. If the quadrics in $(kQ \oplus \mathfrak{q}) \setminus kQ$ have rank at least three, then $F_1 \cap F_2 = \{Q'\}$, where $Q' = \mathcal{V}(f)$ for some $f \in S_1 a + S_1 b$, and Q' corresponds to a nonsingular point on at least one of the cubic divisors F_1 and F_2.*

Proof. (a) The result follows from combining Lemma 1.6 and Proposition 2.2.

(b) By Lemma 1.8, F_1 (respectively, F_2) parametrizes the quadrics in $\mathbb{P}(\mathfrak{Q}/(\mathfrak{Q} \cap S_1 a))$ (respectively, in $\mathbb{P}(\mathfrak{Q}/(\mathfrak{Q} \cap S_1 b))$) of rank at most two (where we have abused notation by treating $\mathfrak{Q} \subset S_2$). Since $kQ \oplus \mathfrak{q}$ is base-point free, $\dim(F_i) = 1$ for $i = 1, 2$ (by Lemma 1.8), and any rank-one quadric in $kQ \oplus \mathfrak{q}$ corresponds to a rank-one quadratic form in S_2 that has a nonzero image in $S/\langle a \rangle$, and similarly in $S/\langle b \rangle$; since these images have rank one, (b) follows.

(c) The first statement follows from (a) and the fact that $kQ \oplus \mathfrak{q}$ is base-point free. As in the proof of (b), $\dim(F_i) = 1$ for $i = 1, 2$. Suppose $F_1 \cap F_2 = \{Q'\}$ and that Q' corresponds to a singular point on both F_1 and F_2, and write $Q' = \mathcal{V}(f)$, where $f \in S_1 a + S_1 b$. It follows that the image of f in $S_2/S_1 a$ has rank one, and similarly for the image of f in $S_2/S_1 b$. Since

$kQ \oplus \mathfrak{q}$ contains no rank-one elements, we have $f \in k^\times a^2 + kab + k^\times b^2$, which contradicts $(kQ \oplus \mathfrak{q}) \setminus kQ$ containing no rank-two elements. ■

The reader should note that the converse of Proposition 2.5(c) is false; that is, if $F_1 \cap F_2 = \{\mathcal{V}(ac + bd)\}$ where $c, d \in S_1$, a, b and c are linearly independent and $d \in ka + kb + k^\times c$ (in particular, $\mathcal{V}(ac + bd)$ corresponds to a nonsingular point on F_1 and F_2), then, by Proposition 2.2(c), $(kQ \oplus \mathfrak{q}) \setminus kQ$ contains a rank-two element.

Recall from Section 1 that a Clifford quantum $\mathbb{P}^3$ has $r_1 + 2r_2$ point modules. The following result enables examples to be constructed of Clifford quantum $\mathbb{P}^3$s with finitely many points.

Theorem 2.6. *Let A denote a Clifford quantum $\mathbb{P}^3$ and let $\mathfrak{Q}$ denote the corresponding quadric system that is base-point free. Suppose $\mathfrak{Q} = kQ \oplus \mathfrak{q}$, where $\mathfrak{q}$ denotes a net of quadrics in $\mathfrak{Q}$ and $Q = \mathcal{V}(ab) \notin \mathfrak{q}$, such that $a, b \in S_1$ are linearly independent. Write F_1 and F_2 for the two planar cubic divisors in $\mathbb{P}(\mathfrak{q})$ parametrizing the quadrics in $\mathbb{P}(\mathfrak{q})$ that meet the two distinct planes of Q in degenerate conics (as in the proof of Lemma 1.6), and suppose that $|F_1 \cap F_2| < \infty$ (that is, $|F_1 \cap F_2| \leq 9$). Let m denote the number of distinct points in $F_1 \cap F_2$, and let r_i denote the number of distinct quadrics in $\mathbb{P}(\mathfrak{Q})$ of rank i, and suppose that $r_1 \leq 1$. We have*

(a) $r_2 < \infty$ *(so, by* [16, Theorem 2.1] *or* [17], $1 \leq r_2 \leq 10$, *and* $r_2 = 10$ *only if* $r_1 = 0$*);*

(b) *A has at most finitely many isomorphism classes of point modules;*

(c) $\dim[\mathfrak{Q} \cap (S_1 a + S_1 b)] \in \{1, 2\}$*;*

(d) $m - r_1 \leq r_2 \leq 2(m - r_1) + 1$ *(however, it will follow from Theorem 3.1 below that* $r_2 \leq \min\{2(m - r_1) + 1,\ 10 - 4r_1\}$*);*

(e) *if each $\mathbb{P}(kQ \oplus kQ')$, where $Q' \in F_1 \cap F_2$, contains a generic number of rank-2 elements, then $r_2 = m + 2 - t - r_1$, where $t = \dim[\mathfrak{Q} \cap (S_1 a + S_1 b)]$.*

Proof.

(a) By construction of F_1 and F_2 in the proof of Lemma 1.6, since $m < \infty$, the only way $\mathbb{P}(\mathfrak{Q})$ can contain infinitely many quadrics of rank two is if there is a quadric $Q' \in F_1 \cap F_2$, such that $\mathbb{P}(kQ \oplus kQ')$ contains infinitely many rank-two elements. By Lemma 2.3, this would mean that $Q' = \mathcal{V}(f)$ where $f \in S_1 a$ or $f \in S_1 b$ or $f \in k^\times a^2 + kab + k^\times b^2$. The first two possibilities contradict the hypothesis that $\mathfrak{Q}$ is base-point free, and the last possibility contradicts the assumption that $r_1 \leq 1$ (since $\mathrm{char}(k) \neq 2$). It follows that $r_2 < \infty$.

(b) As explained in Section 1, A has $r_1 + 2r_2$ isomorphism classes of point modules, so the result follows from (a).

(c) By hypothesis, $\dim[\mathfrak{Q} \cap (S_1 a + S_1 b)] \geq 1$. On the other hand, if

$$\dim[\mathfrak{Q} \cap (S_1 a + S_1 b)] \geq 3,$$

then $\mathfrak{Q}$ would have base points, which is false.

(d) By construction of F_1 and F_2 in the proof of Lemma 1.6, it suffices to count the number of rank-two quadrics obtained by linearly combining elements of $F_1 \cap F_2$ with Q. By (c), there are at most two quadrics in $\mathbb{P}(\mathfrak{Q})$ which belong to $S_1 a + S_1 b$; one of them is Q and, by Proposition 2.2 and Corollary 2.4, the other will yield at most one rank-two quadric (besides Q) when linearly combined with Q, with the generic case being to yield none. By Proposition 2.2 and the proof of Lemma 2.3, if $r_1 = 0$, then each of the other quadrics in $F_1 \cap F_2$ yields one or two rank-two quadrics (besides Q) when linearly combined with Q, with the generic case being to yield one. Including Q, the number of rank-two quadrics obtained in this way is at least m and is at most $1 + 2m$. If $r_1 = 1$, then, by Corollary 2.4, one quadric is lost from the minimal value and two quadrics are lost from the maximal value, giving at least $m - 1$, and at most $2m - 1$, rank-two quadrics.

(e) The definition of t in (e) and the proof of (d) prove (e). ■

The main application of Theorem 2.6 is in constructing examples of Clifford quantum $\mathbb{P}^3$s with a prespecified number of points. This idea will be demonstrated in Section 5.

3. Nonexistence of Certain Clifford Quantum $\mathbb{P}^3$s

By [16, Theorem 2.1] or [17], the maximal finite number of points a Clifford quantum $\mathbb{P}^3$ can have is twenty. In this section, we will prove that there are no Clifford quantum $\mathbb{P}^3$s with exactly n distinct points, where $n \in \{2, 15, 17, 19\}$. For all other integers $n \leq 20$, we have found examples of Clifford quantum $\mathbb{P}^3$s with exactly n distinct points and some are presented in Section 5.

3.1. **The Case of an Odd Number of Points.**

In Section 5, we construct an example of a Clifford quantum $\mathbb{P}^3$ with exactly thirteen distinct points; the next result shows that thirteen is the largest finite odd number that can occur.

Theorem 3.1. *A regular Clifford algebra of global dimension four that has an odd, finite number of distinct points has at most thirteen distinct points.*

Proof. If a Clifford quantum $\mathbb{P}^3$ has exactly an odd number of distinct points, then $r_1 = 1$ and $r_2 < \infty$. Thus, we must prove that $r_2 \leq 6$. If $r_2 \leq 2$, then the algebra has at most five distinct points and satisfies the result; so we may assume $r_2 \geq 3$. Thus, we may write the corresponding quadric system as $\mathfrak{Q} = kQ_4 \oplus \cdots \oplus kQ_1$, where $\operatorname{rank}(Q_1) = 1$ and $\operatorname{rank}(Q_i) = 2$ for all $i = 2, 3, 4$. We write $Q_4 = \mathcal{V}(a_1 a_2)$, where $a_1, a_2 \in S_1$ are linearly independent, and write $\mathfrak{q} = kQ_3 \oplus kQ_2 \oplus kQ_1$. As above, let F_1 and F_2 denote the two planar cubic divisors parametrizing the quadrics in $\mathbb{P}(\mathfrak{q})$ that meet the two distinct planes of Q_4 in degenerate conics (as in the proof of Lemma 1.6). By Lemma 1.8, $F_i \neq \mathbb{P}(\mathfrak{q})$, for $i = 1, 2$.

By Proposition 2.5(b), Q_1 corresponds to a singular point p on both cubic divisors F_1 and F_2, so, for each $i = 1, 2$, the multiplicity, $m_p(F_i)$, of p on F_i is at least two. By [7, Sections 3.3 and 5.1], the intersection multiplicity, $m_p(F_1 \cap F_2)$, of p on $F_1 \cap F_2$ is at least $m_p(F_1)m_p(F_2) \geq 4$.

On the other hand, as explained in Subsection 1.3, we may coordinatize $\mathbb{P}(\mathfrak{Q})$ and consider the projection morphism $(W, X, Y, Z) \mapsto (X, Y, Z) \in \mathbb{P}(\mathfrak{q})$. Let $G \subset \mathbb{P}(\mathfrak{Q})$ denote the scheme of quadrics in $\mathbb{P}(\mathfrak{Q})$ of rank at most two. Under the above projection morphism, by Lemma 1.8, $G \setminus \{Q_4\}$ maps to $F_1 \cap F_2$. Since $r_2 < \infty$, we may apply Corollary 2.4 to $\mathbb{P}(kQ_1 \oplus kQ_4)$ to conclude that $\mathbb{P}(kQ_1 \oplus kQ_4)$ contains only two quadrics of rank at most two (namely, Q_1 and Q_4). So the preimage in $G \setminus \{Q_4\}$ of p is $\{Q_1\}$. It thus follows from the definition of the projection map that the multiplicity of Q_1 on G equals $m_p(F_1 \cap F_2) \geq 4$. Since $\deg(G) = 10$ and $|G| < \infty$, we have $r_2 \leq 10 - 4 = 6$. ■

3.2. **Geometric Interlude.**

The proof that there are no Clifford quantum $\mathbb{P}^3$s with exactly two points utilizes certain geometric results involving the intersection of planar cubic divisors. Those results are presented in this subsection. We will use R to denote the polynomial ring on three generators of degree one, so that $\operatorname{Proj} R = \mathbb{P}^2$.

Lemma 3.2. *Let C, K and L, respectively, denote a nondegenerate conic in $\mathbb{P}^2$, a cubic divisor in $\mathbb{P}^2$ and a line in $\mathbb{P}^2$.*

(a) *If C meets K at exactly one point $p \in \mathbb{P}^2$, then $K = \mathcal{V}(g^3 + Gh)$, where $g, G \in R_1$, $h \in R_2$, $C = \mathcal{V}(h)$ and $\mathcal{V}(g)$ is tangent to C at p.*

(b) *Suppose L meets C at two distinct points. If $C \cup L$ meets K at exactly one point $p \in \mathbb{P}^2$, then p is singular on $C \cup L$ and on K.*

Proof. (a) Choose $z \in R_1$ such that $\mathcal{V}(z)$ is the tangent line to C at p. It follows that there exist $x, y \in R_1$ such that $p = \mathcal{V}(y, z)$ and $C = \mathcal{V}(xz + y^2)$. Since k is algebraically closed, and C is nondegenerate, we have that x, y and z are linearly independent, so $R = k[x, y, z]$. Writing $K = \mathcal{V}(f)$, for some nonzero $f \in R_3$, we must prove that $f \in k^\times z^3 + R_1(xz + y^2)$. Since $f(p) = 0$, $f \in R_2 y + R_2 z$. By hypothesis, $C \cap K = \{p\}$, so $C \cap K \cap (\mathbb{P}^2 \setminus \mathcal{V}(z)) = \emptyset$. On $\mathbb{P}^2 \setminus \mathcal{V}(z)$, $C = \{(-y^2, y, 1) : y \in k\}$. It follows that the equation $f(-y^2, y, 1) = 0$ has no solution, which implies that $f(-y^2, y, 1) \in k^\times \subset k[y]$. This fact, combined with a computation using an arbitrary form of $f \in R_2 y + R_2 z$ evaluated at $(-y^2, y, 1)$, establishes that $f \in k^\times z^3 + R_1(xz + y^2)$.

(b) As in (a), we may assume $R = k[x, y, z]$, $C = \mathcal{V}(xz + y^2)$ and $p = (1, 0, 0)$. Clearly, p is singular on $C \cup L$. Since $C \cap K = \{p\}$, it follows, by (a), that $K = \mathcal{V}(f)$, where $f \in k^\times z^3 + R_1(xz + y^2)$. Moreover, since $p \in L$ and since L meets C at exactly two distinct points, $L = \mathcal{V}(H)$, where $H = y - \beta z$ for some $\beta \in k$. Since $L \cap K = \{p\}$, the equation $f(x, \beta z, z) = 0$ has a unique solution, namely $(x, z) = (1, 0)$, so it follows that $f \in k^\times z^3 + kH(xz + y^2)$. Hence, p is singular on K. ■

Lemma 3.3. *Write $R = k[x, y, z]$ and let $f \in k^\times y^3 + k^\times x^2 z + R_1 z^2 + R_1 yz$.*

(a) *If C is a conic in $\mathbb{P}^2$ that meets $\mathcal{V}(f)$ at exactly one point, $p = (1, 0, 0)$, then $C = \mathcal{V}(z^2)$.*

(b) *If K is a cubic divisor in $\mathbb{P}^2$ that meets $\mathcal{V}(f)$ at exactly one point, $p = (1, 0, 0)$, then $K = \mathcal{V}(g)$ where $g \in k^\times z^3 + kf$.*

Proof. Clearly, the divisor $\mathcal{V}(f)$ in $\mathbb{P}^2$ is nonsingular at p, $\mathcal{V}(z, f) = \{p\}$, and the line $\mathcal{V}(z)$ is the unique tangent line to $V(f)$ at p.

(a) If C is nondegenerate, then one may use Lemma 3.2(a) to rewrite f and prove $\mathcal{V}(z) \subset C$, which is a contradiction. Hence C is degenerate, and (a) follows.

(b) Write $K = \mathcal{V}(g)$ where $g \in R_3$. If $g = zh$ for some $h \in R_2$, then $\mathcal{V}(f, h) = \{p\}$. By (a), $h \in k^\times z^2$, so (b) holds. Thus, we may assume $z \nmid g$. Since $g(p) = 0$, we may write $g = zg_1 + yg_2$ for some $g_1, g_2 \in R_2$ such that $0 \neq g_2 \in kx^2 + kxy + ky^2$. On $\mathbb{P}^2 \setminus \mathcal{V}(z)$, $\mathcal{V}(f, g) = \emptyset$. Hence, the system

$$f(x, y, 1) = 0 = g(x, y, 1) \qquad (*)$$

has no solution. Since the x-degree of $f(x, y, 1)$ is two and the coefficient of x^2 in $f(x, y, 1)$ is a nonzero scalar, we may use the equation $f(x, y, 1) = 0$ to solve for x, and so substitute for x in $g(x, y, 1) = 0$. However, the y-degree of $f(x, y, 1)$ is three, and $\deg(g_1) \leq 2$, so it follows that $(*)$ has no solution only if $g_2 \in ky^2$. Thus, $g \in k^\times y^3 + zg_1$, so there exists $\beta \in k^\times$ such that $g - \beta f = zh$ for some $h \in R_2$ such that $\mathcal{V}(f, h) = \{p\}$. By (a), $h \in k^\times z^2$, which concludes the proof. ■

3.3. **The Case of Two Points.**

In this subsection, we prove that there are no Clifford quantum $\mathbb{P}^3$s with exactly two distinct points. The method of proof is to assume that such an algebra exists and to derive a contradiction. Appearing in the proof are Clifford quantum $\mathbb{P}^1$s and Clifford quantum $\mathbb{P}^2$s. By [2], a Clifford quantum $\mathbb{P}^1$ has one defining relation and a Clifford quantum $\mathbb{P}^2$ has three defining relations. We first prove a lemma that describes Clifford quantum $\mathbb{P}^1$s.

Lemma 3.4. *An algebra* $A = k\langle x, y\rangle/\langle xy + yx - \alpha x^2 - \beta y^2\rangle$, *where* α, $\beta \in k$, *is a regular Clifford algebra of global dimension two if and only if* $\alpha\beta \neq 1$, *and all regular Clifford algebras of global dimension two may be written in this form. Moreover, if one is free to re-choose* x (*respectively, both* x *and* y), *then one may take* $\beta = 0$ (*respectively,* $\alpha = 0 = \beta$).

Proof. By Section 1, any regular Clifford algebra of global dimension two can be written in the given form. The quadric system associated to such an algebra has no base points if and only if $\alpha\beta \neq 1$. The last statement is straightforward and left to the reader. ■

Theorem 3.5. *There do not exist any regular Clifford algebras of global dimension four that have exactly two distinct point modules.*

Proof. If the result is false, then there exists a regular Clifford algebra A of global dimension four that has exactly two distinct point modules. This implies that the base-point-free quadric system $\mathfrak{Q}$ associated to A contains no rank-one quadrics, and that $\mathbb{P}(\mathfrak{Q})$ contains only one rank-two quadric Q. We may write $\mathfrak{Q} = kQ \oplus \mathfrak{q}$, where $\mathfrak{q}$ is a net of quadrics in $\mathfrak{Q}$ such that $Q \notin \mathfrak{q}$. Write $Q = \mathcal{V}(ab)$, where a, $b \in S_1$ are linearly independent, and write F_1 and F_2 for the two cubic divisors in $\mathbb{P}(\mathfrak{q})$ parametrizing the quadrics in $\mathbb{P}(\mathfrak{q})$ that meet the two distinct planes of Q in degenerate conics (as in the proof of Lemma 1.6). At times, in this proof, we will abuse notation by treating $\mathfrak{Q} \subset S_2$.

By Proposition 2.5(c), $F_1 \cap F_2$ contains only one quadric Q', and $Q' = \mathcal{V}(q')$, where $q' \in S_1 a + S_1 b$, and Q' corresponds to a point $p \in \mathbb{P}^2$ which is nonsingular on at least one of the cubic divisors F_1 and F_2. Combining this with the fact that $\mathfrak{Q}$ is base-point free, it follows that

$$\dim\left(\frac{\mathfrak{Q}}{(S_1 a + S_1 b) \cap \mathfrak{Q}}\right) = 2.$$

In particular, by Proposition 1.9, the quadric system $\mathfrak{Q}/((S_1 a + S_1 b) \cap \mathfrak{Q})$ corresponds to a regular Clifford subalgebra D of A of global dimension two. Similarly, by Corollary 1.10,

$$\frac{\mathfrak{Q}}{S_1 a \cap \mathfrak{Q}} \quad \text{and} \quad \frac{\mathfrak{Q}}{S_1 b \cap \mathfrak{Q}}$$

are base-point-free quadric systems and correspond to regular Clifford subalgebras B and C of A, respectively, each of global dimension three, and, by Proposition 1.9, $D \subset B \cap C$.

Since $\mathfrak{Q}$ is base-point free, we may choose $c \in S_1$ such that a, b, c are linearly independent with $q' \in S_1^{\times} a + bc$. It follows that

$$\dim\left(\frac{\mathfrak{Q}}{(S_1 a + S_1 c) \cap \mathfrak{Q}}\right) = 2,$$

so, by Proposition 1.9, $\mathfrak{Q}/((S_1 a + S_1 c) \cap \mathfrak{Q})$ corresponds to a regular Clifford subalgebra E of A of global dimension two.

Choose $y \in D_1 \subset A_1$ such that $y \in (ka + kb + kc)^{\perp}$. By Lemma 3.4, there exists $x \in D_1 = (ka + kb)^{\perp} \setminus ky$ such that, after rescaling y (if necessary), $xy + yx = \lambda_1 x^2$, where $\lambda_1 \in \{0, 1\}$. Similarly, there exists $z \in E_1 = (ka + kc)^{\perp} \setminus ky$ such that $yz + zy = \alpha z^2$, where $\alpha \in k$.

However, B is generated by $(ka)^{\perp}$, so B is generated by x, y, z, and has three defining relations, two of which are $xy + yx = \lambda_1 x^2$ and $yz + zy = \alpha z^2$. The third defining relation of B does not entail $xz + zx$ since $bc \in \mathfrak{Q}/(S_1 a \cap \mathfrak{Q})$. Hence, we may take the third defining relation of B to be an element of $k^{\times} x^2 + k y^2 + k^{\times} z^2$. By rescaling z, if necessary, we find that

$$B = \frac{k\langle x, y, z\rangle}{\langle xy + yx - \lambda_1 x^2,\ yz + zy - \alpha z^2,\ x^2 + \mu y^2 + z^2\rangle},$$

for some $\mu \in k$.

The subalgebra C is generated by $(kb)^{\perp} = kx \oplus ky \oplus kw$ for some $w \in (kb \oplus kc)^{\perp}$. Since $\mathfrak{Q}$ is base-point free and $Q = \mathcal{V}(ab)$ is the only rank-two quadric in $\mathbb{P}(\mathfrak{Q})$, we have that $ac \notin \mathfrak{Q}/(S_1 b \cap \mathfrak{Q})$ and $wx + xw \in kx^2 + ky^2 + kw^2$. We may re-choose w, if necessary, to give $wx + xw = \beta_1 w^2 + \beta_2 y^2$, where $\beta_1, \beta_2 \in k$. Moreover, by changing the choice of c within $ka \oplus kc$, we may continue to assume $w \in (kb \oplus kc)^{\perp}$ and continue to use the established relations.

Thus, we may write the third defining relation of C as $\lambda_2(wy+yw)=\gamma_1 w^2+\gamma_2 x^2+\gamma_3 y^2$, where $\gamma_1,\ \gamma_2,\ \gamma_3 \in k,\ \lambda_2 \in \{0,1\}$. The regularity of D allows $\gamma_1 = 1$ if $\lambda_2 = 0$. Hence, $C = k[x,y,w]$ with defining relations

$$xy+yx=\lambda_1 x^2,\ wx+xw=\beta_1 w^2+\beta_2 y^2,\ \lambda_2(wy+yw)=\gamma_1 w^2+\gamma_2 x^2+\gamma_3 y^2.$$

By rescaling a, b, c, if necessary, there exists $d \in S_1$ such that $\{a, c, d, b\}$ may be used as a dual basis to $\{w, x, y, z\}$.

Dualizing the relations of A and B in turn yields that the quadric systems corresponding to A and B respectively have basis elements:

$$bc, \quad c^2+\lambda_1 cd - b^2 - \alpha bd, \quad d^2 - \mu b^2 - \alpha\mu bd, \tag{1}$$

and

$$\begin{gathered} \lambda_2(a^2+\beta_1 ac)+\gamma_1 ad, \\ c^2+\lambda_1 cd+(\lambda_2-1)\gamma_2(a^2+\beta_1 ac)+\lambda_2\gamma_2 ad, \\ d^2+(1-\lambda_2)((\beta_2-\beta_1\gamma_3)ac-\gamma_3 a^2)+\lambda_2(\beta_2 ac+\gamma_3 ad), \end{gathered} \tag{2}$$

where $\lambda_1,\ \lambda_2,\ \beta_1,\ \beta_2,\ \gamma_1,\ \gamma_2,\ \gamma_3 \in k$ and, as above, $\lambda_1(\lambda_1-1)=0=\lambda_2(\lambda_2-1)$; and if $\lambda_2=0$, we may assume that $\gamma_1=1$.

Lifting these quadric systems to obtain $\mathfrak{Q}$, we find that, after rescaling a and w if necessary, $\mathfrak{Q}$ has basis:

$$\begin{gathered} \{\ ab, \qquad \lambda_2(a^2+\beta_1 ac)+\gamma_1 ad+bc, \\ c^2+\lambda_1 cd+(\lambda_2-1)\gamma_2(a^2+\beta_1 ac)+\lambda_2\gamma_2 ad-b^2-\alpha bd, \\ d^2+(1-\lambda_2)((\beta_2-\beta_1\gamma_3)ac-\gamma_3 a^2)+\lambda_2(\beta_2 ac+\gamma_3 ad)-\mu b^2-\alpha\mu bd\ \}, \end{gathered} \tag{3}$$

where $\lambda_1,\ \lambda_2,\ \beta_1,\ \beta_2,\ \gamma_1,\ \gamma_2,\ \gamma_3 \in k$ and, as above, $\lambda_1(\lambda_1-1)=0=\lambda_2(\lambda_2-1)$; and if $\lambda_2=0$, we may assume that $\gamma_1=1$. We remark that the last three basis elements of (3) span the net $\mathfrak{q}$.

We will derive a contradiction by proving that $|F_1 \cap F_2| \geq 2$.

Let $q \in \mathbb{P}(\mathfrak{Q})$ and write q as a linear combination of the basis elements in (3) with corresponding coefficients W, X, Y, Z, so that q may be represented by $(W, X, Y, Z) \in \mathbb{P}^3 = \mathbb{P}(\mathfrak{Q})$. Recalling Subsection 1.3, the image of q in $\mathbb{P}(\mathfrak{Q}/(\mathfrak{Q}\cap S_1 a))$ is a linear combination of the images of the basis elements given in (3), and hence is a linear combination, with coefficients X, Y, Z, of the basis elements given in (1). Thus, the image of q in $\mathbb{P}(\mathfrak{Q}/(\mathfrak{Q}\cap S_1 a))$ is represented by $(X, Y, Z) \in \mathbb{P}^2$, where $(W, X, Y, Z) \in \mathbb{P}^3$ represents q. Similarly, using the basis in (2), the image of q in $\mathbb{P}(\mathfrak{Q}/(\mathfrak{Q}\cap S_1 b))$ is represented by $(X, Y, Z) \in \mathbb{P}^2$, where $(W, X, Y, Z) \in \mathbb{P}^3$ represents q. By Lemma 1.8, F_1 parametrizes the quadrics in $\mathbb{P}(\mathfrak{Q}/(\mathfrak{Q}\cap S_1 a))$ of rank at most two, and F_2 parametrizes the quadrics in $\mathbb{P}(\mathfrak{Q}/(\mathfrak{Q} \cap S_1 b))$ of rank at most two, so we may use the bases (1) and (2) and the projection map onto the last three coordinates to compute F_1 and F_2.

Using (1), we find $F_1 = \mathcal{V}(f_1)$, where $f_1 \in k[X,Y,Z] = R$ is the determinant of the matrix

$$\begin{bmatrix} -2(Y+\mu Z) & X & -\alpha(Y+\mu Z) \\ X & 2Y & \lambda_1 Y \\ -\alpha(Y+\mu Z) & \lambda_1 Y & 2Z \end{bmatrix}.$$

Hence, $f_1 \in k^\times((\alpha^2 - \lambda_1^2)Y^3 + \alpha\lambda_1 XY^2 + X^2 Z) + R_1 YZ$, where λ_1, α, $\mu \in k$ and $\lambda_1(\lambda_1 - 1) = 0$. The point $p = (1,0,0) \in F_1$, p is nonsingular on F_1 and p corresponds to the quadric $Q' = \mathcal{V}(q')$, where

$$q' = \lambda_2(a^2 + \beta_1 ac) + \gamma_1 ad + bc \in \mathbb{P}(\mathfrak{q}).$$

Similarly, using (2), we find that $F_2 = \mathcal{V}(f_2)$, where $f_2 \in k[X,Y,Z] = R$ is the determinant of the matrix

$$\begin{bmatrix} 2(\lambda_2 X + (\lambda_2 - 1)(\gamma_2 Y + \gamma_3 Z)) & \begin{matrix}\lambda_2(\beta_1 X + \beta_2 Z)+ \\ (\lambda_2 - 1)(\beta_1\gamma_2 Y + (\beta_1\gamma_3 - \beta_2)Z)\end{matrix} & \gamma_1 X + \lambda_2(\gamma_2 Y + \gamma_3 Z) \\ \begin{matrix}\lambda_2(\beta_1 X + \beta_2 Z)+ \\ (\lambda_2 - 1)(\beta_1\gamma_2 Y + (\beta_1\gamma_3 - \beta_2)Z)\end{matrix} & 2Y & \lambda_1 Y \\ \gamma_1 X + \lambda_2(\gamma_2 Y + \gamma_3 Z) & \lambda_1 Y & 2Z \end{bmatrix}.$$

In particular, $p \in F_2$.

Case 1: $\lambda_2 = 0$.

Since $\lambda_2 = 0$, we may assume that $\gamma_1 = 1$. Thus,

$$f_2 \in k^\times(\beta_2 - \beta_1\gamma_3)^2 Z^3 + k^\times X^2 Y + R_1 YZ + R_1 Y^2,$$

so p is nonsingular on F_2. If $\beta_2 \neq \beta_1\gamma_3$, then we may apply Lemma 3.3(b) to f_2, so that the $X^2 Z$-term in f_1 implies that $|F_1 \cap F_2| \geq 2$. However, if $\beta_2 = \beta_1\gamma_3$, then $Y | f_2$ and, since p is nonsingular on F_2, we again have $|F_1 \cap F_2| \geq 2$.

Case 2: $\lambda_2 = 1$ and $\lambda_1 = 0$.

If $\alpha = 0$, then $Z | f_1$, and, since p is nonsingular on F_1, we have $|F_1 \cap F_2| \geq 2$. If $\alpha \neq 0$, but $\gamma_1 = 0$, then there is an XYZ-term in f_2, so applying Lemma 3.3(b) to f_1 implies that $|F_1 \cap F_2| \geq 2$. On the other hand, if $\alpha\gamma_1 \neq 0$, then we may again apply Lemma 3.3(b) to f_1, and the presence of an $X^2 Y$-term in f_2 implies $|F_1 \cap F_2| \geq 2$.

Case 3: $\lambda_2 = 1 = \lambda_1$.

If $\beta_2 = 0$, then $(0,0,1) \in F_1 \cap F_2$, so $|F_1 \cap F_2| \geq 2$. Hence, we may assume $\beta_2 \neq 0$. If $\alpha = 0$, then we may apply Lemma 3.3(b) to f_1, which implies that $f_2 \in kf_1 + k^\times Z^3$. Thus, there is no XZ^2-term in f_2 and so $\beta_1 = 0$. This, in turn, implies that there is no X^2Z-term in f_2, and so $f_2 \in k^\times Z^3$. It follows that $\gamma_2 = 0$ (to eliminate the Y^3-term from f_2), but that forces $f_2 \notin k^\times Z^3$, which is a contradiction. Hence, we may assume that $\alpha\beta_2 \neq 0$.

Suppose that $\beta_1 = 0 = \gamma_1$. If $\alpha^2 - 1 \neq \alpha\gamma_2^2$, then $kf_1 + kf_2$ contains a nonzero polynomial f belonging to $k^\times Y^3 + X^2Z + R_1Z^2 + R_1YZ$. Applying Lemma 3.3(b) to f, we find that to have $|F_1 \cap F_2| = 1$, we require that $f_i \in kf + k^\times Z^3$ for $i = 1, 2$. This is false due to the XY^2-terms in f_1 and f_2. On the other hand, if $\alpha^2 - 1 = \alpha\gamma_2^2$, then $kf_1 + kf_2$ contains a nonzero reducible polynomial f (divisible by Z) and p is nonsingular on $\mathcal{V}(f)$, so again $|F_1 \cap F_2| \geq 2$.

Suppose that $\beta_1 = 0$ and $\gamma_1 \neq 0$. Reversing the roles of y and z in Lemma 3.3(b), we may apply Lemma 3.3(b) to f_2, so that we require $f_1 \in kf_2 + k^\times Y^3$ in order to have $|F_1 \cap F_2| = 1$. However, there is an X^2Y-term in f_2 but not in f_1, so we require $f_1 \in k^\times Y^3$. Since this is false, we have $|F_1 \cap F_2| \geq 2$.

At this stage, we may assume $\alpha\beta_1\beta_2 \neq 0$. In this setting,

$$f_1 \in k^\times[(\alpha^2-1)Y^3 + \alpha XY^2 + X^2Z + \alpha\mu XYZ + (2\alpha^2\mu - \mu + 4)Y^2Z + (\alpha^2\mu^2 + 4\mu)YZ^2]$$

and

$$\begin{aligned} f_2 \in k^\times[-\gamma_2^2Y^3 + (\beta_1\gamma_2 - 2\gamma_1\gamma_2 - 1)XY^2 + \gamma_1(\beta_1 - \gamma_1)X^2Y + \\ - \beta_1^2X^2Z - 2\beta_1\beta_2XZ^2 - \beta_2^2Z^3 + (4 + \beta_2\gamma_1 + \beta_1\gamma_3 - 2\gamma_1\gamma_3)XYZ + \\ + \gamma_2(\beta_2 - 2\gamma_3)Y^2Z + \gamma_3(\beta_2 - \gamma_3)YZ^2]. \end{aligned}$$

Suppose that it is possible to change basis so that $X \mapsto X$, $Y \mapsto Y$ and $Z \mapsto$ an element (also called Z) of $k^\times Z + kY$ to obtain an element $f \in kf_1 + kf_2$ with an X^2Z-term, but no X^2Y-term nor XY^2-term. Since p is nonsingular on $\mathcal{V}(f)$, if $Z|f$, then $|F_1 \cap F_2| \geq 2$. On the other hand, if $Z \nmid f$, then f has a nonzero Y^3-term, so we may apply Lemma 3.3(b) to f, which forces $f_1 \in kf + k^\times Z^3$ in order to have $|F_1 \cap F_2| = 1$. However, either f_1 has gained an X^2Y-term from the basis change so that $f_1 \notin kf + k^\times Z^3$, or there was no basis change and so there is an XY^2-term in f_1, so that again $f_1 \notin kf + k^\times Z^3$. In either case, $|F_1 \cap F_2| \geq 2$.

Suppose that it is not possible to change basis as described above to obtain an element $f \in kf_1 + kf_2$ with an X^2Z-term, but no X^2Y-term nor XY^2-term. Let $\nu \in k$. Since

$$f_2 + \nu f_1 = X^2[(\nu - \beta_1^2)Z + \gamma_1(\beta_1 - \gamma_1)Y] + \dots,$$

it follows that, after the basis change given by

$$X \mapsto X, \quad Y \mapsto Y \quad \text{and} \quad Z \mapsto (\nu - \beta_1^2)Z + \gamma_1(\beta_1 - \gamma_1)Y,$$

the failure to obtain f of the desired form is caused by $\nu = \beta_1^2$ being the only value of ν which, after the above basis change, yields zero for the coefficient of XY^2 in $f_2 + \nu f_1$. Analysis of the expression giving that coefficient shows that its numerator equals $\alpha(\nu - \beta_1^2)^3$. However, the numerator may also be written in terms of $\alpha,\ \mu \ldots, \gamma_3$, and equating those two polynomials one obtains an equation that holds for all $\nu \in k$. Given that $\alpha\beta_1\beta_2 \neq 0$, it follows that one cannot perform the desired basis change if and only if

$$\gamma_1(\gamma_1 - \beta_1) = 0 \quad \text{and} \quad \alpha\beta_1^2 = 1 - \gamma_2(\beta_1 - 2\gamma_1). \tag{4}$$

Hence, if (4) holds, we cannot perform a basis change as described above, but there exists a nonzero $f \in (kf_1 + kf_2) \cap (kY^3 + k^\times XZ^2 + k^\times Z^3 + R_1 YZ)$. If $Z|f$, then, since $|F_1 \cap \mathcal{V}(Z)| \geq 2$, we have $|F_1 \cap F_2| \geq 2$. Thus, we may assume that

$$0 \neq f \in (kf_1 + kf_2) \cap (k^\times Y^3 + k^\times XZ^2 + k^\times Z^3 + R_1 YZ),$$

and that $\gamma_2^2 \neq \beta_1^2(\alpha^2 - 1)$.

Since $\mathcal{V}(f_1, f_2) = \{p\}$, the system of equations

$$f_1(X, Y, 1) = 0 = f(X, Y, 1) \tag{5}$$

has no solution. One may solve (5) by using $f_1(X, Y, 1) = 0$ to solve for X, and then substituting the solution into $f(X, Y, 1) = 0$. Doing so yields an equation $g = 0$, where $g = \sum_{i=0}^{6} g_i Y^i \in k[Y]$, with the $g_i \in k$ being functions of the seven parameters α, μ, β_1, β_2, γ_1, γ_2 and γ_3; moreover, $g_0 \in k^\times \beta_2^2$, so $g_0 \neq 0$. Since (5) has no solution, we have $g = g_0$ and $g_i = 0$ for $i = 1, \ldots, 6$. Combining this data with (4) gives eight equations to be satisfied by the seven parameters. The constraint $\alpha\beta_1\beta_2 \neq 0$, together with the fact that $Z \nmid f$, causes this system of eight equations to have no solution (see Remark 3.6 below). It follows that $\deg(g) > 0$ and so (5) has a solution after all. Hence, $|F_1 \cap F_2| \geq 2$. ∎

Remark 3.6. Solving the eight equations that appear towards the end of the proof of Theorem 3.5 appears to be nontrivial, even with the use of current sophisticated computer-algebra software. Instead, it is recommended that one first rewrite the coefficients of f as, say, δ_j, and then solve the equations $g_i = 0$, $i = 3, \ldots 6$, for the δ_j as functions of the coefficient, δ, of XYZ in f. Substituting those solutions into $g_1 = 0 = g_2$ yields solutions for μ and δ in terms of α. Some of these solutions imply that the Z^3-coefficient in f is zero, which is false. Using each of the remaining solutions for (μ, δ) with (4) and with the δ_i written in terms of $\alpha, \ldots, \gamma_3$, one can eliminate four of the seven variables $\alpha, \ldots, \gamma_3$. With computer-algebra software, one can then find a Gröbner basis for the four polynomials that are the numerators of the remaining expressions.

4. Regular Clifford Algebras with an Odd Number of Points

In this section, we show that if A is a regular Clifford algebra of global dimension $d \geq 2$ with at most a finite number, n, of isomorphism classes of point modules, then n is odd if and only if A is an Ore extension of a regular Clifford subalgebra of global dimension $d-1$. We conclude this section by classifying Clifford quantum $\mathbb{P}^2$s up to isomorphism.

4.1. **Ore Extensions.**

Lemma 4.1. *Let R denote a quadratic regular algebra, ρ a graded automorphism of R, and δ a graded ρ-derivation of degree one. If A is the Ore extension, $A = R[w; \rho, \delta]$, then $M = A/R_1A$ is a point module over A, and $M[1]_{\geq 0} \cong M$.*

Proof. The module M is isomorphic, as a vector space, to $k[w]$, with R_1 acting as zero and w acting in the natural way. ■

Remark 4.2. By [16, Proposition 1.5 and Lemma 1.6], a point module M over a regular Clifford algebra, A, such that $M[1]_{\geq 0} \cong M$, corresponds to a rank-one element of the quadric system associated to A, and conversely.

Lemma 4.3. *Let R denote a regular Clifford algebra, and let ρ denote the graded automorphism of R induced by multiplication by -1 on R_1. If δ is a graded ρ-derivation of R of degree one such that, for all $r \in R_1$, $\delta(r)$ belongs to the space of symmetric elements in R_2, then the Ore extension $R[w; \rho, \delta]$ is a regular Clifford algebra.*

Proof. The algebra $R[w; \rho, \delta]$ is quadratic with symmetric relations by construction, and it is regular by [11]. The result follows. ■

Proposition 4.4. *Let R denote a regular Clifford algebra, and let ρ denote the graded automorphism of R induced by multiplication by -1 on R_1. Suppose δ is a graded ρ-derivation of R of degree one such that, for all $r \in R_1$, $\delta(r)$ belongs to the space of symmetric elements in R_2, and let $A = R[w; \rho, \delta]$. If the number, n, of distinct isomorphism classes of point modules over A is finite, then n is odd.*

Proof. By Lemma 4.3, A is a regular Clifford algebra. Recall from Subsection 1.1 that if $\mathfrak{Q}$ denotes the quadric system associated to A, then A has exactly $r_1 + 2r_2$ distinct isomorphism classes of point modules, where r_i denotes the number of elements in $\mathbb{P}(\mathfrak{Q})$ of rank i. By Lemma 4.1 and Remark 4.2, $r_1 \geq 1$, and, by [16], if $r_1 + 2r_2 < \infty$, then $r_1 \leq 1$. ∎

Proposition 4.5. *If A denotes a regular Clifford algebra of global dimension $d \geq 2$, then the following are equivalent:*

(a) *the algebra A is an Ore extension of a regular Clifford algebra of global dimension $d-1$;*

(b) *there exists a point module M over A such that $M[1]_{\geq 0} \cong M$;*

(c) *there exists $w \in A_1$ such that w^2 does not occur in any defining relation of A; that is, A has a presentation as a quadratic algebra $T(V^*)/\langle W \rangle$, where $V^* = A_1$ and $W \subseteq V^* \otimes_k V^*$, such that no $w \otimes w$-term occurs in any element of W.*

Proof. By [4, Section 1], A has d generators and $d(d-1)/2$ defining relations, and has the same Hilbert series as the polynomial ring in d variables, namely $H_A(t) = 1/(1-t)^d$. Let $\mathfrak{Q}$ denote the linear system of quadrics associated to A.

Statement (a) implies (b), by Lemma 4.1.

To prove (b) implies (c), let M denote a point module over A such that $M[1]_{\geq 0} \cong M$. By Remark 4.2, M corresponds to a rank-one quadric in $\mathbb{P}(\mathfrak{Q})$; in particular, $\mathfrak{Q}$ contains a rank-one element $\mathcal{V}(\eta^2)$, for some $\eta \in A_1^*$. Let $w \in A_1 \setminus (k\eta)^\perp$. Since $w\eta \in k^\times$, and $\mathcal{V}(\eta^2) \in \mathfrak{Q}$, it follows that w^2 does not occur in any defining relation of A. Hence, (b) implies (c).

Suppose that (c) holds. The hypothesis on w implies that $\mathfrak{Q}$ contains a rank-one quadric $\mathcal{V}(\eta^2)$ where $\eta \in A_1^* \setminus (kw)^\perp$. By Corollary 1.10, A contains a regular Clifford subalgebra B of global dimension $d-1$, and the linear system of quadrics associated to B is base-point free and is given by $\mathfrak{Q}/(S_1\eta \cap \mathfrak{Q})$ (where we have abused notation by treating $\mathfrak{Q} \subset S_2$). It follows that $w \notin B$.

By comparing the Hilbert series of A with that of B, we find that A is a free B-module. We define an automorphism $\rho : B \to B$ via $\rho(b) = -b$ for all $b \in B_1$ and a map $\delta : B \to A$ by $\delta(b) = wb - \rho(b)w$ for all $b \in B$. Since A is a regular Clifford algebra and since w^2 does not occur in any defining relation of A, it follows, from Definition 1.2, that $wb + bw \in B$ for all

$b \in B_1$, so that $\delta(b) \in B$ for all $b \in B_1$. Moreover, if $b, b' \in B$, then

$$\begin{aligned} \delta(bb') &= wbb' - \rho(bb')w \\ &= (\rho(b)w + \delta(b))b' - \rho(b)\rho(b')w \\ &= \rho(b)[\rho(b')w + \delta(b')] + \delta(b)b' - \rho(b)\rho(b')w \\ &= \rho(b)\delta(b') + \delta(b)b'. \end{aligned}$$

It follows, by induction, that $\delta(B) \subset B$ and that δ is a ρ-derivation of B. Thus $B[w; \rho, \delta]$ is an Ore extension of B that is contained in A and has the same Hilbert series as A. Hence, $A = B[w; \rho, \delta]$. ■

Theorem 4.6. *If A is a regular Clifford algebra of global dimension $d \geq 2$ with exactly n distinct isomorphism classes of point modules, where $n < \infty$, then n is odd if and only if A is an Ore extension of a regular Clifford algebra of global dimension $d - 1$.*

Proof. If A is an Ore extension of a regular Clifford algebra of global dimension $d-1$, then, by Proposition 4.5, A has a point module M such that $M[1]_{\geq 0} \cong M$. By Remark 4.2, M corresponds to a rank-one element in the linear system $\mathfrak{Q}$ of quadrics associated to A. The fact that n is odd now follows from the proof of Proposition 4.4.

On the other hand, if n is odd, then $\mathfrak{Q}$ contains a rank-one element Q, and, by Remark 4.2, the point module M corresponding to Q satisfies $M[1]_{\geq 0} \cong M$. The result follows from Proposition 4.5. ■

4.2. **Clifford Quantum $\mathbb{P}^2$s.**

By Theorem 4.6, if a Clifford quantum $\mathbb{P}^3$ has exactly an odd number of distinct points, then the algebra is an Ore extension of a Clifford quantum $\mathbb{P}^2$. As such, we classify Clifford quantum $\mathbb{P}^2$s, up to isomorphism, in this subsection.

Lemma 4.7. *Let A denote a Clifford quantum $\mathbb{P}^2$, and let σ denote the automorphism encoded in the point scheme of A. If the point scheme of A contains a fixed point of σ, then A is an Ore extension of a Clifford quantum $\mathbb{P}^1$, and $A \cong k[x, y, z]$ with defining relations*

$$xy + yx = 0, \qquad xz + zx = \alpha y^2, \qquad yz + zy = \beta x^2,$$

where $\alpha, \beta \in \{0, 1\}$.

Proof. The existence of a fixed point of σ implies that A has a point module M with the property that $M[1]_{\geq 1} \cong M$, so we may apply Proposition 4.5 to obtain that A is an Ore extension of a Clifford quantum $\mathbb{P}^1$. By Lemma 3.4, the Clifford quantum $\mathbb{P}^1$ is isomorphic to

$k[x, y]$ with the defining relation $xy + yx = 0$. Hence, we may write A as an Ore extension of this subalgebra by a variable z with defining relations

$$xy + yx = 0, \qquad xz + zx = \alpha_1 x^2 + \alpha_2 y^2, \qquad yz + zy = \beta_1 x^2 + \beta_2 y^2,$$

for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in k$. Since

$$x\left(z - \frac{\alpha_1}{2}x - \frac{\beta_2}{2}y\right) + \left(z - \frac{\alpha_1}{2}x - \frac{\beta_2}{2}y\right)x = \alpha_2 y^2$$

and

$$y\left(z - \frac{\alpha_1}{2}x - \frac{\beta_2}{2}y\right) + \left(z - \frac{\alpha_1}{2}x - \frac{\beta_2}{2}y\right)y = \beta_1 x^2,$$

we may change z in order to rewrite the defining relations of A as

$$xy + yx = 0, \qquad xz + zx = \alpha y^2, \qquad yz + zy = \beta x^2,$$

for some $\alpha, \beta \in k$. By rescaling x or y, if necessary, we may take $\alpha, \beta \in \{0, 1\}$. ■

Corollary 4.8. *If A is a Clifford quantum $\mathbb{P}^2$, then $A \cong k[x, y, z]$ with defining relations*

$$xy + yx = \lambda_1 z^2, \qquad xz + zx = \lambda_2 y^2, \qquad yz + zy = \lambda_3 x^2,$$

where

$$(\lambda_1, \lambda_2, \lambda_3) \in \{(0, 0, 0),\ (0, 0, 1),\ (0, 1, 1),\ (\lambda, \lambda, \lambda)\ :\ \lambda \in k,\ \lambda^3 \notin \{0, -1, 8\}\}.$$

For each allowed choice of $(\lambda_1, \lambda_2, \lambda_3)$, the point scheme of A is as follows:

- *if $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$, then the point scheme of A is given by the "triangle" $\mathcal{V}(xyz)$;*
- *if $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 1)$, then the point scheme of A is given by the union of a conic and a line: $\mathcal{V}(x(x^2 + 2yz))$;*
- *if $(\lambda_1, \lambda_2, \lambda_3) = (0, 1, 1)$, then the point scheme of A is given by the nodal cubic curve $\mathcal{V}(x^3 + y^3 + 2xyz)$, where the singular point is $(0, 0, 1) \in \mathbb{P}^2$;*
- *if $(\lambda_1, \lambda_2, \lambda_3) = (\lambda, \lambda, \lambda)$, where $\lambda^3 \notin \{0, -1, 8\}$, then the point scheme of A is given by the nonsingular elliptic curve: $\mathcal{V}(\lambda(x^3 + y^3 + z^3) + (2 - \lambda^3)xyz)$.*

Proof. If the point scheme of A contains a fixed point of the encoded automorphism, then the relations of A follow from Lemma 4.7, with $\lambda_1 = 0$ and $\lambda_2, \lambda_3 \in \{0, 1\}$. On the other hand, if the point scheme does not contain a fixed point, then, by [2], the point scheme is a nonsingular elliptic curve, and so the relations follow, in this case, from [2] and symmetry. The second statement is a computation. ■

5. Constructing Clifford Quantum $\mathbb{P}^3$s with a Prespecified Number of Points

In this section, we focus on methods for constructing explicit examples of Clifford quantum $\mathbb{P}^3$s with n distinct points, where $n < \infty$ and prespecified. The reader is referred to Section 3 for the proof that $n \in \{1, 3, 4, 5, \ldots, 13, 14, 16, 18, 20\} = X$. For each $n' \in X$, we have found Clifford quantum $\mathbb{P}^3$s with $n = n'$, but, since the discussion of only a few of these examples illustrates the key ideas in the construction of all our examples, only a few will be presented.

By Theorem 4.6, if $n \in X$ is odd, then one may construct a Clifford quantum $\mathbb{P}^3$ with n distinct points as an Ore extension of a Clifford quantum $\mathbb{P}^2$. Although such Ore extensions are discussed in detail in Section 4 and the notion of Ore extension is well understood, this method has at least two shortcomings: firstly, it is unclear how to control the value of n to be finite, and, secondly, it is unclear how to force n to be equal to any prespecified odd value in X. Nevertheless, with the aid of a computer, one can find examples via this method ([6]).

Alternatively, if $n \in X$, then one may construct a Clifford quantum $\mathbb{P}^3$ with n distinct points by using the two planar cubic divisors introduced in Section 1. This method is illustrated below in Example 5.3, and can be used for any value of $n \in X$, but is perhaps most powerful if n is small. This method has the advantage that the points so produced typically have integral or rational coordinates, and so help keep further computations manageable.

There is a third method that is useful if n is somewhat large (e.g., $n \geq 14$), and is illustrated in our initial examples below. It can be used in conjunction with the two planar cubic divisors.

5.1. **An Example with Twenty Points.**

By Section 1, to construct a Clifford quantum $\mathbb{P}^3$ with exactly twenty distinct points, we must find a base-point-free web $\mathfrak{Q}$ of quadrics in $\mathbb{P}^3$ such that $\mathbb{P}(\mathfrak{Q})$ contains exactly ten rank-two quadrics. The key idea we use here is Lemma 2.3 in order to maximize the finite number of rank-two quadrics in $\mathbb{P}(\mathfrak{Q})$.

Suppose that $a, b, c, d \in S_1$ are linearly independent and let $\mathfrak{q} = k(a^2 - b^2) \oplus k(a^2 - c^2) \oplus k(a^2 - d^2)$. We find that $\mathbb{P}(\mathfrak{q})$ contains exactly six rank-two quadrics (and none of rank one) which is the maximal finite number of rank-two quadrics any projectivized net of quadrics can contain (by Lemma 2.3). In the spirit of Lemma 2.3, we set $Q = a^2 - e^2$, for some $e \in S_1$, and define $\mathfrak{Q} = kQ \oplus \mathfrak{q}$. Thus, we must choose $e \in S_1$ such that $Q \notin \mathfrak{q}$, $\mathfrak{Q}$ is base-point free, and such that $\mathbb{P}(\mathfrak{Q})$ contains exactly ten rank-two quadrics. By Lemma 2.3, and the fact that having no base points is an open condition, the collection of such $e \in S_1$ is dense in S_1. Hence, almost any choice of e will satisfy our constraints. In particular, if we take $e = a + b + c + d$, then $\mathfrak{Q}$ is a base-point-free web of quadrics (if $\text{char}(k) \neq 3$ or 5) such that $\mathbb{P}(\mathfrak{Q})$ contains exactly ten rank-two quadrics (if $\text{char}(k) = 3$ or 5, then choose a different e). One can check this by standard methods, but it is simpler to check it using the two cubic divisors. In this case, one could set $a = \pm b$ and find the corresponding two cubic divisors and their six intersection points (see Theorem 2.6), and then count the number of rank-two quadrics obtained by linearly combining the quadrics corresponding to the intersection points with $a^2 - b^2$.

An advantage of the approach outlined above is that there is a choice of homogeneous coordinates on $\mathbb{P}^3$ such that the twenty distinct points so found have coordinates in $\mathbb{Z}$, which can help simplify any further computations desired with this algebra.

5.2. **An Example with Thirteen Points.**

By Section 1, to construct a Clifford quantum $\mathbb{P}^3$ with exactly thirteen distinct points, we must find a base-point-free web $\mathfrak{Q}$ of quadrics in $\mathbb{P}^3$ such that $\mathbb{P}(\mathfrak{Q})$ contains exactly six rank-two quadrics and only one rank-one quadric. We follow the method of Subsection 5.1, using the same $\mathfrak{q}$, but setting $Q = e^2$, for some $e \in S_1$. By the arguments in Subsection 5.1, the collection of $e \in S_1$ that give a web $\mathfrak{Q} = kQ \oplus \mathfrak{q}$ satisfying the necessary conditions is dense in S_1. In particular, if we take $e = a - b + c + 2d$, then e satisfies our constraints if $\text{char}(k) \neq 3$ or 5 (if $\text{char}(k) = 3$ or 5, then choose a different e). As in Subsection 5.1, it is straightforward to check the result, and the thirteen distinct points so obtained can be written with coordinates in $\mathbb{Z}$.

5.3. **An Example with Three Points.**

Constructing a Clifford quantum $\mathbb{P}^3$ with a small number of points is more challenging than the cases presented in Subsections 5.1 and 5.2. Since examples are known with one point ([16]), and since there are none with two points (by Theorem 3.5), we construct an example, A, with three points. Thus, we wish to find a base-point-free web $\mathfrak{Q}$ of quadrics in $\mathbb{P}^3$ such that $\mathbb{P}(\mathfrak{Q})$ contains exactly one rank-two quadric and exactly one rank-one quadric.

Let $a, b, c, d \in S_1$ be linearly independent and fix the rank-two element $ab \in S_2$ and the rank-one element $c^2 \in S_2$. With the goal of minimizing the number of rank-two elements, we also fix an element in $S_1 a + S_1 b$ (see Proposition 2.2). There are many choices for such an element, but here we choose $ad + b(c+d) \in S_1 a + S_1 b$, and define $\mathfrak{q} = kab \oplus kc^2 \oplus k(ad + bc + bd)$.

At this stage, it is unclear how to choose the fourth basis element, Q, of $\mathfrak{Q} = kQ \oplus \mathfrak{q}$ (where we have abused notation by identifying $\mathfrak{Q}$ with its counterpart in S_2), but, by solving for base points, we find that we must take $Q = \alpha_1 a^2 + \alpha_2 b^2 + \alpha_3 d^2 + \ldots$, for some $\alpha_1, \alpha_2, \alpha_3 \in k^\times$.

Following the proof of Theorem 3.5, we observe that each of the quadric systems

$$\frac{\mathfrak{Q}}{S_1 a \cap \mathfrak{Q}}, \qquad \frac{\mathfrak{Q}}{S_1 b \cap \mathfrak{Q}} \qquad \text{and} \qquad \frac{\mathfrak{Q}}{S_1 c \cap \mathfrak{Q}}$$

is base-point free and corresponds to a regular Clifford subalgebra of A (by Corollary 1.10) of global dimension three. Similarly, by Proposition 1.9,

$$\frac{\mathfrak{Q}}{(S_1 a + S_1 c) \cap \mathfrak{Q}}, \qquad \frac{\mathfrak{Q}}{(S_1 b + S_1 c) \cap \mathfrak{Q}} \qquad \text{and} \qquad \frac{\mathfrak{Q}}{(S_1 d + S_1 c) \cap \mathfrak{Q}}$$

are base-point-free quadric systems and correspond to regular Clifford subalgebras of A of global dimension two. Using $\{w, x, y, z\}$ as a dual basis to $\{a, c, d, b\}$, the latter quadric system implies the relation $\alpha_1 z^2 = \alpha_2 w^2$ in A. Similarly, from $\mathfrak{Q}/((S_1 b + S_1 c) \cap \mathfrak{Q})$, we obtain $\alpha_1 y^2 = \alpha_3 w^2$ in A. This gives two of the three relations determining the Clifford quantum $\mathbb{P}^2$ corresponding to $\mathfrak{Q}/(S_1 c \cap \mathfrak{Q})$; it follows that its third relation is $yz + zy = yw + wy$. By Section 4, A is an Ore extension by x of this Clifford quantum $\mathbb{P}^2$.

At this stage, we have that $Q = \alpha_1 a^2 + \alpha_2 b^2 + \alpha_3 d^2 + c(\ldots)$. The defining relations of the algebra corresponding to $\mathfrak{Q}/(S_1 a \cap \mathfrak{Q})$ are

$$\alpha_2 y^2 = \alpha_3 z^2, \qquad xz + zx = yz + zy \qquad \text{and} \qquad xy + yx = (\beta/\alpha_3) y^2$$

for some $\beta \in k$. It follows that $Q = \alpha_1 a^2 + \alpha_2 b^2 + \alpha_3 d^2 + \beta cd + c(\ldots)$. Repeating this argument for $\mathfrak{Q}/(S_1 b \cap \mathfrak{Q})$, we obtain

$$Q = \alpha_1 a^2 + \alpha_2 b^2 + \alpha_3 d^2 + \beta cd + \gamma ac$$

for some $\gamma \in k$. With this choice of Q, the web $\mathfrak{Q}$ is base-point free for all $\alpha_1, \alpha_2, \alpha_3 \in k^\times$, $\beta, \gamma \in k$. We must find values for the parameters $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ such that $\mathbb{P}(\mathfrak{Q})$ contains exactly one rank-two element (namely, ab) and exactly one rank-one element (namely, c^2). To find such values we may use the planar cubic divisors F_1 and F_2 associated to $\mathfrak{Q}/(S_1 a \cap \mathfrak{Q})$ and $\mathfrak{Q}/(S_1 b \cap \mathfrak{Q})$, respectively (see Theorem 2.6). Since the third basis element $ad + bc + bd \in (S_1 a + S_1 b) \cap \mathfrak{Q}$, by Theorem 2.6, we require $|F_1 \cap F_2| = 2$.

By writing out the 3×3 matrices that determine the F_i, one can guess at values to try for the parameters. In particular, renaming $\alpha_1 = \alpha$ and setting $\alpha_2 = \alpha_1$, $\alpha_3 = 1$, $\beta = 0$, the relevant 3×3 matrices determining F_1 and F_2 are

$$\begin{bmatrix} 2\alpha Z & Y & Y \\ Y & 2X & 0 \\ Y & 0 & 2Z \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 2\alpha Z & \gamma Z & Y \\ \gamma Z & 2X & 0 \\ Y & 0 & 2Z \end{bmatrix}$$

respectively, and their determinants (up to nonzero scalar multiples) are

$$f_1 = XY^2 - 4\alpha XZ^2 + Y^2Z \qquad \text{and} \qquad f_2 = XY^2 - 4\alpha XZ^2 + 4\gamma^2 Z^3$$

respectively, where $F_1 = \mathcal{V}(f_1)$ and $F_2 = \mathcal{V}(f_2)$. Intersecting F_1 and F_2, we find that if we additionally assume that $\gamma^2 = 4\alpha$, then the intersection points are $(1, 0, 0)$ and $(0, 1, 0)$ in $\mathbb{P}^2$. These points correspond to the elements c^2 and $ad + bc + bd$ respectively, neither of which linearly combines with ab to yield a new element of rank at most two. It follows that

$$\mathfrak{Q} = kab \oplus kc^2 \oplus k(ad + bc + bd) \oplus k(\gamma^2 a^2 + \gamma^2 b^2 + 4d^2 + 4\gamma ac),$$

where $\gamma \in k^\times$, is a web of quadrics in $\mathbb{P}^3$ such that $\mathbb{P}(\mathfrak{Q})$ contains exactly one rank-two element (namely, ab) and exactly one rank-one element (namely, c^2). Hence, the Clifford quantum $\mathbb{P}^3$ corresponding to $\mathfrak{Q}$ has exactly three points, and these points can be written with their coordinates in $\{0, 1\}$.

Some further examples of finite Clifford quantum $\mathbb{P}^3$s are given in [6].

In a sequel to this article, we will deform the relations of finite Clifford quantum $\mathbb{P}^3$s to produce finite quantum $\mathbb{P}^3$s that are not Clifford algebras, and which have a one-dimensional line scheme.

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