Four-dimensional Regular Algebras with Point Scheme a Nonsingular Quadric in $\mathbb{P}^3$

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Abstract. In [22], a class of four-dimensional, quadratic, Artin-Schelter regular algebras was introduced, whose point scheme is the graph of an automorphism of a nonsingular quadric in $\mathbb{P}^3$. These algebras are the first examples of quadratic Artin-Schelter regular algebras whose defining relations are not determined by the point scheme and, hence, not determined by the algebraic data obtained from the point modules. In this paper, we study these algebras via their line modules. In particular, the set of lines in $\mathbb{P}^3$ that correspond to left line modules is not the set of lines in $\mathbb{P}^3$ that correspond to right line modules. Our analysis focuses on a distinguished member R_λ of this class of algebras, where R_λ is a twist by a twisting system of the other algebras. We prove that R_λ is a finite module over its center and that its central Proj is a smooth quadric in $\mathbb{P}^4$.

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Introduction

Throughout the paper, let k denote an algebraically closed field such that $\mathrm{char}(k) \neq 2$.

In the non-commutative projective geometry developed by M. Artin in [2], a quantum $\mathbb{P}^2$ is defined as the Proj of an Artin-Schelter regular algebra of global dimension three which is generated by degree-one elements. Such algebras were classified by M. Artin, J. Tate and M. Van den Bergh in [4, 5] where certain "non-commutative geometric" techniques were used. A quadratic algebra giving a quantum $\mathbb{P}^2$ has the same Hilbert series as the polynomial ring on three generators and has point scheme which is the graph of an automorphism of a subscheme of $\mathbb{P}^2$; moreover, such an algebra is determined by its point scheme in the following sense.

Definition. [22] Let P be a subscheme of $\mathbb{P}^n$ and $\sigma \in \mathrm{Aut}(P)$. The quadratic algebra determined by the geometric data (P, σ) is the graded algebra

$$\frac{T(U)}{\langle f \in U \otimes U : \; f|_{\Gamma_\sigma} = 0 \rangle}$$

where U is an $(n+1)$-dimensional vector space, $T(U)$ is the tensor algebra on U, $\mathbb{P}(U^*)$ is identified with the copy of $\mathbb{P}^n$ containing P and $\Gamma_\sigma \subset \mathbb{P}(U^*) \times \mathbb{P}(U^*)$ is the graph of σ.

In this article, we take a "quantum $\mathbb{P}^3$" to be the Proj of a quadratic, Artin-Schelter regular algebra A of global dimension four which has the same Hilbert series as the polynomial ring on four generators. The geometric methods of [4, 5] were extended in [22, Theorem 1.10] and [15, Theorem 1.4] to those A satisfying certain homological conditions; namely, for such A, if $\mathbb{P}(A_1^*)$ is identified with $\mathbb{P}^3$, then A has point scheme which is the graph Γ of an automorphism σ of a subscheme P of $\mathbb{P}^3$. It follows from [4] that there is a bijective correspondence between Γ and the isomorphism classes of left (respectively right) point modules over A. Moreover, Γ has the property that it is the zero locus in $\mathbb{P}^3 \times \mathbb{P}^3$ of the defining relations of A.

The geometric methods used in [4, 5] have been applied to many quadratic Artin-Schelter regular algebras of global dimension four, such as the examples in [10, 11, 12, 13, 17, 20, 21, 23]. However, there is at present little hope of obtaining a full classification of regular algebras of global dimension four along the lines of [4, 5]. The main difficulty is that, in contrast to the lower dimensional case, the point scheme (P, σ) need not determine the quadratic algebra.

The first examples of quadratic, four-dimensional, regular algebras R whose point scheme (P, σ) determines an algebra nonisomorphic to R were found in [22, 23]. Such algebras R are part of the classification in [22] of those algebras whose Proj, a quantum $\mathbb{P}^3$, embeds, in some sense, a quantum nonsingular quadric; that is, those algebras A which have the property that they map onto a twisted homogeneous coordinate ring of a nonsingular quadric $Q \subset \mathbb{P}^3$. The scheme P from the point scheme (P, σ) of A is either Q itself, or $\mathbb{P}^3$, or the union of Q and a line in $\mathbb{P}^3$. In the first case, where $P = Q$, the point scheme (P, σ) determines a quotient of A, not the algebra A itself; this is the case investigated herein.

Each algebra from the previous paragraph for which $P = Q$ has the property that it is twisting-system equivalent to a certain algebra R_λ where $\lambda \in k$ ([22, Proposition 4.12]). As such, the graded representation theory and, in particular, the projective geometry of such algebras may be derived from that of R_λ, which is defined in Section 1. The point scheme of R_λ is (Q, τ) for all λ, and so it does not distinguish between the R_λ.

In Section 2 we show that the scalar λ is encoded in the description of the line modules, so that, unlike the point scheme, the scheme of line modules distinguishes between the R_λ. This is related to the fact that, in contrast to the known quadratic regular algebras of dimension four, the set of lines in $\mathbb{P}^3$ corresponding to the left line modules differs from the set of lines in $\mathbb{P}^3$ corresponding to the right line modules, c.f., Corollary 2.4. This is the case in spite of the fact that, by [13], there is a one-to-one correspondence between these sets of lines.

In Section 3 we study the non-commutative projective geometry from a more classical point of view, showing that R_λ is a free module of rank sixteen over its center Z_λ.

In Section 4 we study $\mathcal{R}_\lambda$, the structure sheaf of R_λ over $\operatorname{Proj} Z_\lambda = \mathbb{P}^3$, and the central Proj, $\mathbf{Spec}\mathcal{Z}$, which is determined by the center $\mathcal{Z}$ of $\mathcal{R}_\lambda$. The main results of Section 4 are Theorems 4.10 and 4.12, which may be summarized as follows.

> The sheaf $\mathcal{R}_\lambda$ is a sheaf of quaternion algebras over its center $\mathcal{Z}$.
> The central Proj, $\mathbf{Spec}\mathcal{Z}$, is a smooth quadric in $\mathbb{P}^4$ and thus determines a degree-two cover of $\mathbb{P}^3 = \operatorname{Proj} Z_\lambda$.

1. The Algebras R_λ with Point Scheme a Nonsingular Quadric in $\mathbb{P}^3$

By [22, Proposition 4.12], the algebras in [22] whose point scheme is (Q, τ), where Q is a nonsingular quadric in $\mathbb{P}^3$ and $\tau \in \operatorname{Aut}(Q)$, may be classified up to isomorphism and up to twisting-system equivalence by a one-parameter subfamily $\{A(\alpha, \tau)\}$, where $\alpha \in k$, $\alpha(\alpha^2 + 1) \neq 0$. Recall that by construction $A(\alpha, \tau)$ maps onto the twisted homogeneous coordinate ring S_τ of Q with respect to the automorphism $\tau \in \operatorname{Aut}(Q)$; in fact, $S_\tau = A(\alpha, \tau)/\langle \Omega \rangle$ where Ω is the unique (up to scalar multiples) homogeneous, degree-two element of $A(\alpha, \tau)$ which vanishes on the point scheme of $A(\alpha, \tau)$. By [22], the element Ω is normal in $A(\alpha, \tau)$. The algebra $A(\alpha, \tau)$ is not determined by its point scheme (Q, τ), (in the sense of the definition given in the Introduction), since this geometric data determines S_τ.

In [22], generators $x_1, \ldots, x_4$ are chosen for $A(\alpha, \tau)$ such that

$$Q = \mathcal{V}(x_1x_4 - x_2x_3) \subset \mathbb{P}^3 \qquad \text{and} \qquad \tau = \begin{pmatrix} 1 & 1 & i & i \\ 1 & 1 & -i & -i \\ -i & i & 1 & -1 \\ -i & i & -1 & 1 \end{pmatrix}.$$

In this paper, a different choice of generators is used as follows.

Lemma 1.1. *The algebra $A(\alpha, \tau)$ is isomorphic to the algebra $R_\lambda = k[u, v, x, y]$ with defining relations*

$$\begin{array}{rclcrcl} ux - xu & = & 0, & \qquad & vy + yv & = & 0, \\ uy - yu & = & 0, & & uv + vu & = & \lambda(xy - yx), \\ vx + xv & = & 0, & & xy + yx & = & u^2 + v^2, \end{array}$$

where $\lambda \in k$, $\lambda(\lambda^2 - 1) \neq 0$. With respect to this choice of generators, the point scheme (Q, τ) of R_λ is given by

$$Q = \mathcal{V}(v^2 - u^2 + 2xy) \subset \mathbb{P}^3 \qquad \textit{and} \qquad \tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. If u, v, x, $y \in A(\alpha, \tau)_1$ are defined by

$$\begin{array}{ll} u = x_1 + x_2 - i(x_3 - x_4), & \qquad x = -2i(x_1 + ix_3), \\ v = x_1 - x_2 - i(x_3 + x_4), & \qquad y = i(x_2 - ix_4), \end{array}$$

where $i \in k$, $i^2 = -1$, and if $\lambda = (\alpha - i)/(\alpha + i) \in k$, then u, v, x and y are linearly independent and satisfy the above relations. It follows that $A(\alpha, \tau)$ is isomorphic to R_λ as claimed and that $Q = \mathcal{V}(v^2 - u^2 + 2xy)$. The matrix representing τ with respect to u, v, x and y, which are eigenvectors of τ, follows since u, x, and y have eigenvalue 1, whereas v has eigenvalue -1. ■

We remark that there is a symmetry between the variables u, v (respectively, antisymmetry between x, y) in the defining relations of R_λ. Furthermore, it is shown in [24, Proposition 4.4.13] that $R_\lambda \cong R_{\lambda'}$ if and only if $\lambda' \in \{\pm\lambda, \pm 1/\lambda\}$.

The following result is derived immediately from the defining relations of R_λ.

Lemma 1.2.

(a) *The elements* u^2, v^2, x^2 *and* y^2 *are central in* R_λ.

(b) *The element* $uv - vu$ *is normal and skewcommutes with all homogeneous elements of odd degree in* R_λ.

(c) *In* R_λ *the normal element* Ω *is* $xy - yx$. *Moreover, as in* [22], Ω *determines an automorphism* θ *of* R_λ *via:* $\Omega r = \theta(r)\Omega$ *for all* $r \in R_\lambda$. *In particular,* $\theta(u) = u$, $\theta(v) = v$, $\theta(x) = -x$ *and* $\theta(y) = -y$.

The center of R_λ is determined in Section 3 to be $k[u^2, v^2, x^2, y^2]$.

2. Line Modules over R_λ

For convenience, we henceforth denote R_λ by R. Unless otherwise stated, modules will be left R-modules. Recall from [4] that a line module over R is a graded, cyclic R-module with Hilbert series $(1-t)^{-2}$. We denote the line module determined by the line ℓ by $M(\ell)$. Let $r^\tau \in R_1$ denote $\tau(r)$ for all $r \in R_1$.

In Theorem 2.2, we prove that the lines through any point $p \in \mathbb{P}^3$ which correspond to a line module over R lie on a plane T_p passing through p; moreover, every line on T_p which passes through p corresponds to a line module over R. In addition, in Remark 2.3, we demonstrate that the affine cone over a line not lying on Q which corresponds to a line module is a two-dimensional isotropic subspace of a certain skew-symmetric bilinear form and all such two-dimensional isotropic subspaces of the bilinear form arise in this way. The bilinear form may be derived from the *line scheme* as shown in [16].

By [13, Proposition 2.8], a line $\mathcal{V}(a, b) \subset \mathbb{P}^3$ determines a left line module over R if and only if $R_1 a \cap R_1 b \neq 0$; in fact, the elements $c, d \in R_1$ which arise from this intersection via $ca = db$ in R are unique up to nonzero scalar multiples. With this notation, we have that if the line $\mathcal{V}(a, b)$ determines a left line module over R, then associated to it is the line $\mathcal{V}(c, d)$ which corresponds to a right line module over R.

The fact that R maps onto S_τ implies that the line modules over S_τ are line modules over R, so the lines on Q determine line modules over R. Any line $\mathcal{V}(a,b)$ in $\mathbb{P}^3$ not lying on Q does not correspond to a line module over S_τ, so $\Omega \notin R_1 a + R_1 b$.

Remark 2.1. Since the normal element Ω is the unique element of R_2, up to nonzero scalar multiples, which vanishes on the graph of τ, we have that $a^\tau b - b^\tau a \in k\Omega$. It follows that if the line $\mathcal{V}(a,b) \not\subset Q$, then it corresponds to a line module over R if and only if $a^\tau b = b^\tau a$ in R.

The following result was proved in [23, Theorem 4.7] for a certain algebra which is isomorphic to a twist (by a twisting system) of R.

Theorem 2.2. *If $p = (u_p, v_p, x_p, y_p) \in \mathbb{P}^3$, then the lines in $\mathbb{P}^3$ which contain p and which correspond to left (respectively, right) line modules over R are the lines that lie in the plane T_p where*

$$T_p = \mathcal{V}(v_p\, u - u_p\, v + \lambda y_p\, x - \lambda x_p\, y)$$

(respectively, $T_p = \mathcal{V}(v_p\, u - u_p\, v - \lambda y_p\, x + \lambda x_p\, y)$). In particular, if $p \in Q$, then T_p is the plane determined by the two lines on Q which pass through p.

Proof. If $p \in Q$, then there are exactly two lines on Q which pass through p since $\operatorname{rank}(Q) = 4$; these two lines correspond to line modules over R since they do over S_τ.

For any $p \in \mathbb{P}^3$, write $p = \mathcal{V}(a_1, a_2, a_3)$ where the $a_i \in R_1$. By Remark 2.1, we have that $a_i^\tau a_j - a_j^\tau a_i = \delta_{ij}\Omega$ for all $i, j \in \{1,2,3\}$, where $\delta_{ij} \in k$ for all i, j.

If $\delta_{ij} = 0 = \delta_{im}$ where $\{i,j,m\} = \{1,2,3\}$, then the two lines $\ell_{ij} = \mathcal{V}(a_i, a_j)$ and $\ell_{im} = \mathcal{V}(a_i, a_m)$ correspond to line modules over R, and so does any line passing through p which lies in the plane T_p spanned by ℓ_{ij} and ℓ_{im}. If, in addition, there is a line through p not lying in T_p which corresponds to a line module over R, then $a_j^\tau a_m = a_m^\tau a_j$ in R, so that every line through p corresponds to a line module over R. In this case, $\dim(R_1 a_1 + R_1 a_2 + R_1 a_3) \leq 9$. Since the point scheme is the graph of an automorphism, it follows that p corresponds to a point module, so $p \in Q$. By rechoosing the a_i if necessary, we find that $a_1^\tau x = x^\tau a_1$ and $a_2^\tau x = x^\tau a_2$ for all $x \in R_1$. It follows that $P \neq Q$, which is false.

On the other hand, if at most one of the δ_{ij} is zero, then the three lines

$$\ell^i_{mj} = \mathcal{V}(a_i, (-1)^j \delta_{ij} a_m + (-1)^m \delta_{im} a_j),$$

where i, j, m cycle through 1, 2, 3, are lines which contain p and correspond to line modules over R since

$$a_i^\tau((-1)^j \delta_{ij} a_m + (-1)^m \delta_{im} a_j) = ((-1)^j \delta_{ij} a_m + (-1)^m \delta_{im} a_j)^\tau a_i$$

in R. If the three lines ℓ^i_{mj} span a plane T_p, then, using the above argument, we find that lines that pass through p and correspond to line modules over R lie in the plane T_p. However, if the three lines ℓ^i_{mj} span more than a plane, then every line through p corresponds to a line module over R. In this case, we may invoke the above argument to conclude that $P \neq Q$, which is false.

Hence, for any $p \in \mathbb{P}^3$, there is a plane T_p which contains p such that every line through p which lies on T_p corresponds to a line module over R and, moreover, any line through p which corresponds to a line module lies on T_p. It remains to find T_p.

Let $a = v_p\, u - u_p\, v + \lambda y_p\, x - \lambda x_p\, y \in R_1$, and consider the plane $\mathcal{V} = \mathcal{V}(a)$ which contains p. It suffices to prove that $\mathcal{V} = T_p$. An arbitrary line in $\mathcal{V}$ which passes through p has the form $\mathcal{V}(a, b)$ where $b = \mu u + \beta v + \gamma x + \delta y$ for some μ, β, γ, $\delta \in k$ and $b(p) = 0$. Using the action of τ and the fact that $b(p) = 0$, it follows that $R_1 a \cap R_1 b \neq 0$, since

$$\begin{aligned} b^\tau a - a^\tau b \;\; = \;\; & (uv + vu - \lambda(xy - yx))(\gamma x_p + \delta y_p) + \\ & \quad + (ux - xu)(\lambda \mu y_p - \gamma v_p) - (uy - yu)(\lambda \mu x_p + \delta v_p) + \\ & \qquad + (yv + vy)(\lambda \beta x_p - \delta u_p) - (xv + vx)(\lambda \beta y_p + \gamma u_p), \end{aligned}$$

which is zero in R for all μ, β, γ, $\delta \in k$. Hence, every line in the plane $\mathcal{V}$ which passes through p corresponds to a left line module over R, and since T_p is the unique plane with this property, $T_p = \mathcal{V}$.

A similar argument applies to right line modules by considering the plane $\mathcal{V}(a)$, where $a = v_p\, u - u_p\, v - \lambda y_p\, x + \lambda x_p\, y$, which contains p. Taking b as above, a computation verifies that $ab^{\tau^{-1}} = ba^{\tau^{-1}}$ in R, so that $aR_1 \cap bR_1 \neq 0$. ■

Remark 2.3. In [16], it is proved that if A is a quadratic algebra on four generators with six defining relations, then there exists a scheme, called the *line scheme*, which represents the functor of truncated left line modules over A of length three, and similarly for right line modules. In our setting, since any line module, $L = \bigoplus_{i\geq 0}^{\infty} L_i$, over R is determined by L_0, L_1 and L_2, it follows that the line scheme of R parametrizes the line modules over R, and hence yields a global description of the line modules.

It is shown in [16] that the line scheme of R consists of three linear components, one of which is three-dimensional, with the other two being one-dimensional and corresponding to the two rulings on the quadric Q. The lines described by the three-dimensional component may be realized as the two-dimensional isotropic subspaces in R_1^* of a certain nondegenerate skew-symmetric bilinear form as follows.

Since a line in $\mathbb{P}^3$ is the image of a two-dimensional subspace of $\mathbb{A}^4$, the Grassmannian $\mathbb{G} = \mathbb{G}(2,4)$ of two-dimensional subspaces of $\mathbb{A}^4$ is the scheme parametrizing the lines in $\mathbb{P}^3$. So a line in $\mathbb{P}^3$ is a point of $\mathbb{G}$. Moreover, it is conventional to represent a line $\ell = \mathbb{P}(kU \oplus kV)$ in $\mathbb{P}^3$, where U, $V \in \mathbb{P}^3$, as a $2{\times}4$ matrix $(U\ V)^T$, and the coordinate variable M_{ij} on $\mathbb{G}$, where $1 \leq i < j \leq 4$, computes the ij'th minor of the 2×4 matrix. As such, the Grassmannian $\mathbb{G}$ may be naturally embedded in $\mathbb{P}^5$ as the subscheme

$$\mathbb{G} = \mathcal{V}(M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}),$$

where the M_{ij}, $1 \le i < j \le 4$, are the coordinates on $\mathbb{P}^5$. In particular, $\mathbb{G}$ is irreducible and $\dim(\mathbb{G}) = 4$. The M_{ij}, $1 \le i < j \le 4$, are called the *Plücker coordinates* on $\mathbb{G}$, and depend on the choice of coordinates used on $\mathbb{P}^3$.

Using the generators of R given in Lemma 1.1, the three-dimensional component of the line scheme of R, which parametrizes the left (respectively, right) line modules that correspond to lines not lying on Q, is $\mathcal{V}(M_{12} + \lambda M_{34}) \subset \mathbb{G}$ (respectively, $\mathcal{V}(M_{12} - \lambda M_{34}) \subset \mathbb{G}$), and so is a hyperplane section of $\mathbb{G}$. It therefore yields a skew-symmetric bilinear form $B : R_1^* \times R_1^* \to k$, where $B = u \wedge v + \lambda x \wedge y$ (respectively, $u \wedge v - \lambda x \wedge y$) which is nondegenerate; that is, with respect to the basis u, v, x, y on R_1,

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & -\lambda & 0 \end{pmatrix} \quad \text{(respectively,} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda \\ 0 & 0 & \lambda & 0 \end{pmatrix} \text{)},$$

which has rank four. The lines in $\mathbb{P}^3 = \mathbb{P}(R_1^*)$ not lying on Q which yield line modules over R correspond to the two-dimensional isotropic subspaces of B in R_1^*; namely, subspaces W of the form

$$W = \{w \in R_1^* : B(w, w') = 0 \text{ for all } w' \in W\}$$

that are two-dimensional.

It follows that the line scheme distinguishes between the nonisomorphic members of the family $\{R_\lambda\}$, which is in contrast to the point scheme (Q, τ), which is the same for all members of the family $\{R_\lambda\}$. For a more detailed discussion of this example, and also of the line scheme of the other algebras appearing in [22], see [16].

By the proof of Theorem 2.2, if a line $\ell \not\subset Q$ corresponds to a left line module over R, then $\tau^{-1}\ell$ corresponds to a right line module over R. However, by [22], the action of τ on R_1 cannot be extended to an algebra automorphism of R, which suggests that the set of lines corresponding to left line modules is not the set of lines corresponding to right line modules.

Corollary 2.4.

(a) *The set of lines in $\mathbb{P}^3$ corresponding to the left line modules over R is not the set of lines in $\mathbb{P}^3$ which correspond to the right line modules over R.*

(b) *The lines in $\mathbb{P}^3$ that correspond to both left line modules and right line modules over R are the lines that either lie on Q or meet both lines $\mathcal{V}(u, v)$ and $\mathcal{V}(x, y)$.*

(c) *The lines that correspond to left line modules but not to right line modules may be characterized as the lines ℓ which meet neither the line $\mathcal{V}(u, v)$ nor the line $\mathcal{V}(x, y)$ and which have the property that if ℓ' is any line which meets the lines ℓ, $\mathcal{V}(u, v)$ and $\mathcal{V}(x, y)$, then the plane determined by ℓ and ℓ' is the plane T_p where $p = \ell \cap \ell'$.*

Note. The lines $\mathcal{V}(u, v)$, $\mathcal{V}(x, y)$, which are the lines of fixed points of the automorphism θ of Lemma 1.2, do not correspond to line modules over R.

Proof.

(a) Let ℓ be a line in $\mathbb{P}^3$, and fix any $p \in \ell$, and write $p = (u_p, v_p, x_p, y_p)$. By Theorem 2.2, ℓ corresponds to a left line module if and only if $\ell \subset \mathcal{V}(v_p\, u - u_p\, v + \lambda y_p\, x - \lambda x_p\, y)$, and ℓ corresponds to a right line module if and only if $\ell \subset \mathcal{V}(v_p\, u - u_p\, v - \lambda y_p\, x + \lambda x_p\, y)$. Since $\lambda \neq 0$, it follows that ℓ corresponds to both a left line module and a right line module if and only if $\ell = \mathcal{V}(v_p\, u - u_p\, v,\ y_p\, x - x_p\, y)$. Part (a) follows.

(b) By the proof of (a), if a line ℓ in $\mathbb{P}^3$ corresponds to both a left line module and a right line module, then ℓ either lies on Q or meets both lines $\mathcal{V}(u, v)$ and $\mathcal{V}(x, y)$. If $p \in \mathcal{V}(u, v)$, then $T_p = \mathcal{V}(p_y x - p_x y)$, which is the plane determined by p and $\mathcal{V}(x, y)$, and similarly if $p \in \mathcal{V}(x, y)$, then T_p is the plane determined by p and $\mathcal{V}(u, v)$. Thus the lines in (b) correspond to both left line modules and right line modules.

(c) If ℓ is a line satisfying the characterization in (c), then ℓ corresponds to a line module since ℓ is contained in some T_p. By (b), the line ℓ corresponds to either a left line module or a right one, but not both. Conversely, if a line ℓ corresponds to only a left line module, then $\ell \not\subset Q$ and ℓ does not meet one of $\mathcal{V}(u,v)$ and $\mathcal{V}(x,y)$. Suppose that ℓ meets $\mathcal{V}(u,v)$ but not $\mathcal{V}(x,y)$, and let $q = \ell \cap \mathcal{V}(u,v)$. Then $\ell \subset T_q$ and, by the proof of (b), T_q is the plane determined by q and $\mathcal{V}(x,y)$. Thus, ℓ meets $\mathcal{V}(x,y)$ which contradicts (b). So ℓ does not meet $\mathcal{V}(u,v)$ and similarly ℓ does not meet $\mathcal{V}(x,y)$. Moreover, if ℓ' is a line that meets ℓ, $\mathcal{V}(u,v)$ and $\mathcal{V}(x,y)$, then, by (b), ℓ' corresponds to a left line module. The result follows from uniqueness of T_p where $p = \ell \cap \ell'$. ■

The reader is referred to [24, Section 5.2] for a classification of the points $p \in \mathbb{P}^3$ for which $T_p \cap Q$ is a degenerate conic.

3. Descriptions of R_λ and its Center

For convenience, we continue to denote R_λ by R. The existence of the central elements u^2, v^2, x^2, y^2 and the defining relations of R suggest that R is close (in some sense) to a graded Clifford algebra, the definition of which is given in Definition 3.2. Although we prove in Proposition 3.3 that R itself is not a Clifford algebra, it is however the quotient of an iterated Ore extension of a Clifford algebra. In this section, we give four different descriptions of R; any one of them yields that R is a domain of p.i.-degree four with center $Z = k[u^2, v^2, x^2, y^2]$. All four approaches are used in Section 4, where we compute the section algebras of the sheaf of algebras $\mathcal{R}_\lambda = \mathcal{R}$ over $\operatorname{Proj} Z$.

Before explicitly calculating the center of R in Theorem 3.8, we first prove some related results.

Proposition 3.1. *The algebra R is a finite module over its center and has no nonzero normal elements of odd degree.*

Proof. Using the Poincaré-Birkhoff-Witt basis for R, it follows that the Hilbert series of $R/\langle u^2, v^2, x^2, y^2\rangle$ is $H(t) = (1+t)^4$, so $\{u^2, v^2, x^2, y^2\}$ is a regular sequence of central elements in R. The graded version of Nakayama's Lemma, namely [14, Lemma I.7.5, Lemma II.1.3], implies that R is a free module of rank sixteen over the commutative subring $k[u^2, v^2, x^2, y^2]$, and hence is a finite module over its center.

Suppose $a \in R$ is a normal element of odd degree which commutes with elements of R by means of an algebra automorphism $\psi \in \mathrm{Aut}(R)$, that is, $ar = r^\psi a$ for all $r \in R$. Since ψ is defined on R, we have that $\psi \circ \tau = \tau \circ \psi$ on Q and $\psi \in \mathrm{Aut}(Q)$. The former implies that ψ is represented by a matrix of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 \\ \alpha_4 & \alpha_5 & \alpha_6 & 0 \\ \alpha_7 & \alpha_8 & \alpha_9 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the $\alpha_i \in k$. However, $uv - vu$ skewcommutes with all elements of degree one, so it skewcommutes with all odd-degree elements, including a. Moreover, a commutes with u^2, v^2, x^2 and y^2. It follows that ψ is of the form

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \alpha_5 & 0 & 0 \\ 0 & 0 & \alpha_9 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\alpha_5^2 = 1$ and $\alpha_9^2 = 1$. Such a matrix fails to determine an algebra automorphism of R, which contradicts the fact that $\psi \in \mathrm{Aut}(Q)$. ■

We recall some standard facts about Clifford algebras. For more details, the reader is referred to [8] and [9, §4]. Let $C = k[c_1, \ldots, c_m]$ denote the commutative polynomial ring on m variables and let $M = (M_{ij}) \in M_n(C)$ denote a symmetric matrix whose entries are homogeneous linear polynomials in the variables c_i.

Definition 3.2. Let C and M be as above.

(a) The Clifford algebra $\mathrm{Cl}(C, M)$ over C associated to M is the k-algebra with generators $x_1, \ldots, x_n, c_1, \ldots, c_m$ and defining relations $x_i x_j + x_j x_i = M_{ij}$ for all $i, j \le n$, and c_r central for all $r \le m$.

(b) A quaternion algebra is a Clifford algebra of rank four over its center.

Declaring $\deg(x_i) = 1$ and $\deg(c_j) = 2$ for all $i \leq n$ and $j \leq m$ defines an $\mathbb{N}$-grading on $\mathrm{Cl}(C, M)$.

Proposition 3.3. *The algebra R is* not *an $\mathbb{N}$-graded Clifford algebra in the above sense.*

Proof. A necessary condition for R to be an $\mathbb{N}$-graded Clifford algebra over a central subring C is that it have defining relations of the form $x_i x_j + x_j x_i = y_{ij}$ where the $x_i \in R_1$ and the $y_{ij} \in C$ are central elements in R which have degree two in R. In this case, x_i^2 would be central in R for all i. A computation verifies that no elements $x_i, x_j \in R_1$, which have x_i^2 and x_j^2 central in R, yield an element $x_i x_j + x_j x_i$ which is central in R. ■

Writing $M = M_1 c_1 + \ldots + M_m c_m$ where the $M_i \in M_n(k)$ are symmetric matrices, we may associate to the matrix M an m-dimensional linear system $\mathcal{Q} = kQ_1 + \ldots + kQ_m$ of quadrics $Q_1, \ldots Q_m \subset \mathbb{P}^{n-1}$ by taking each Q_i to be the quadric in $\mathbb{P}^{n-1}$ corresponding to M_i. A base point of $\mathcal{Q}$ is a common point of intersection of all the Q_i.

The following results will be used in the sequel.

Proposition 3.4. [9, Proposition 7] *The Clifford algebra $Cl(C, M)$ is a quadratic, Auslander-regular algebra of global dimension n satisfying the Cohen-Macaulay property if and only if the system $\mathcal{Q} = kQ_1 + \ldots + kQ_n$ has no base points.*

Proposition 3.5. [8, Chapter 5] *Let $Cl(C, M)$ be a Clifford algebra as in Definition 3.2, and let K be the field of fractions of C.*

(a) *If the quadratic form, which sends $x \in C^n$ to $x^T M x$, is regular, then the Clifford algebra $Cl(K, M)$ is the ring of quotients of $Cl(C, M)$.*

(b) *If $n \in 2\mathbb{N}$, then $Cl(K, M)$ is a central, simple K-algebra of dimension 2^n.*

(c) *If $n \in 2\mathbb{N} - 1$ and if $\delta_M = (-1)^{\frac{n(n-1)}{2}} \det(M) \notin (K^\times)^2$, then $Cl(K, M)$ is a central, simple algebra over the field $K(\sqrt{\delta_M})$ of dimension 2^{n-1}.*

(d) *If* $n \in 2\mathbb{N}-1$ *and if* $\delta_M \in (K^\times)^2$, *then* $Cl(K,M)$ *is the direct sum of two isomorphic central, simple* K*-algebras of dimension* 2^{n-1}.

For the sequel we introduce the notion of ramification locus. If A is an algebra which is a finite module over its center Z, then its ramification locus is the set of $P \in \operatorname{Spec} Z$ such that $\text{p.i.deg}(A/P) < \text{p.i.deg}(A)$. The localizations at points outside the ramification locus are then precisely the Azumaya localizations, c.f., the theorem of Artin and Procesi in [6, §12.6].

By Proposition 3.5, the ramification locus of the Clifford algebra $\text{Cl}(C,M)$ is determined by the symmetric matrix $M \in M_n(C)$ in the following way:

- if $n \in 2\mathbb{N}$, then the determinant of M determines the ramification locus;
- if $n \in 2\mathbb{N}-1$, then the $(n-1)\times(n-1)$ minors of M determine the ramification locus.

Returning to $R = R_\lambda$, let Z denote the central subalgebra $k[u^2,v^2,x^2,y^2]$ of R. We now construct four Clifford subalgebras of R, namely

$$\begin{array}{rclcrcl} S_u &=& Z[v,x,y], && S_x &=& Z[xy-yx,u,v,y], \\ S_v &=& Z[u,x,y], && S_y &=& Z[xy-yx,u,v,x], \end{array}$$

and express R as a quotient of an iterated Ore extension of each of these four subalgebras.

- **Case (a)** The map $S_u[z;\sigma_u,\delta_u] \to R$.

The algebra $S_u = Z[v,x,y]$ with defining relations

$$vx + xv = 0, \qquad vy + yv = 0, \qquad xy + yx = u^2 + v^2,$$

is the Clifford algebra $\text{Cl}(Z, M_u)$ over Z associated to the symmetric matrix

$$M_u = \begin{pmatrix} 2v^2 & 0 & 0 \\ 0 & 2x^2 & u^2+v^2 \\ 0 & u^2+v^2 & 2y^2 \end{pmatrix},$$

where, using the terminology of Definition 3.2, we set $x_1 = v$, $x_2 = x$ and $x_3 = y$. Since the determinant $\det(M_u) = -2v^2(xy-yx)^2$ is not a square in the function field, $K = k(u^2,v^2,x^2,y^2)$, of Z, we deduce from Proposition 3.5

that the ring of fractions of S_u is a quaternion algebra (Definition 3.2) over its center $K(\sqrt{-\det(M_u)}\ )$. Writing

$$w = v(xy - yx) + x(yv - vy) + y(vx - xv),$$

we have $w^2 = -\frac{9}{2}\det(M_u)$ and w is central in S_u.

Lemma 3.6. *The subalgebra S_u is a domain of p.i.-degree two over its center $Z[w]$, which is a quadratic extension of Z.*

Proof. By Proposition 3.4, S_u is Auslander-regular and satisfies the Cohen-Macaulay property. By Stafford's theorem, [19], it follows that S_u is a maximal order in its ring of fractions. Therefore, the center of S_u is the integral closure, $\bar{Z}$, of Z in the center $K(\sqrt{-\det(M_u)}\ )$ of the ring of fractions of S_u; that is, $Z[w] \subseteq$ the center of $S_u = \bar{Z} \subseteq K(w)$. It follows that the center of S_u is $Z[w]$, otherwise $K(w)$ would not be the center of the ring of fractions of S_u. ■

By the defining relations of S_u, the linear map σ_u defined by

$$\sigma_u(u, v, x, y) = (u, -v, x, y),$$

extends to an automorphism of S_u, and the linear map δ_u defined by

$$\delta_u(u, v, x, y) = (0, \lambda(xy - yx), 0, 0),$$

extends to a σ_u-derivation of S_u. Thus the Ore extension $S_u[z; \sigma_u, \delta_u]$ is well defined. Moreover, there is a surjective homomorphism $S_u[z; \sigma_u, \delta_u] \to R$ which is defined by sending z to $u \in R$.

Lemma 3.7. *The Ore extension $S_u[z; \sigma_u, \delta_u]$ is a domain of p.i.-degree four.*

Proof. Consider the localization, $T_u = S_u[w^{-1}]$, of S_u at the central element $w \in S_u$. Since $\det(M_u)$ is a multiple of w^2, the ramification locus of S_u is the zero locus of w, and hence T_u is an Azumaya algebra over its center $Z[w, w^{-1}]$. It is well known (see, for example, [1, Lemma 1.4]), that one may write

$$T_u[z; \sigma_u, \delta_u] = T_u[z'; \sigma_u]$$

where $z' = z + h$ and $h = \delta_u(w)(\sigma_u(w) - w)^{-1} = -\frac{1}{2}\delta_u(w)w^{-1} \in T_u$. The automorphism σ_u is not inner on T_u as it acts nontrivially on the center of T_u (since $\sigma_u(w) = -w$). However, we have $\sigma_u^2 = \mathrm{id}_{T_u}$, and the center of $T_u[z'; \sigma_u]$ is therefore $(Z(T_u))^{\sigma_u}[(z')^2]$, where $Z(T_u)$ denotes the center of T_u. A computation shows that $\delta_u(w) = -12\lambda x^2y^2 + 3\lambda(u^2 + v^2)^2$, which is central in T_u. Hence

$$hz + zh = \frac{\delta_u(w)^2 w^{-2}}{-2}$$

and $(z')^2 \in k[u^2, v^2, x^2, y^2][\det(M_u)^{-1}]$. It follows that the center of $T_u[z'; \sigma_u]$ is

$$(Z(T_u))^{\sigma_u}[(z')^2] = k[u^2, v^2, x^2, y^2][\det(M_u)^{-1}].$$

As T_u is a free module of rank four over $Z(T_u)$, which is of rank two over the fixed ring $Z(T_u)^{\sigma_u}$, we have that $T_u[z'; \sigma_u]$ has rank sixteen over its center and so is a domain of p.i.-degree four. The same result applies to $S_u[z; \sigma_u, \delta_u]$. ■

Using Lemma 3.7, we now prove the main result of Section 3.

Theorem 3.8. *The algebra R is a domain of p.i.-degree four with center $Z = k[u^2, v^2, x^2, y^2]$.*

Proof. We adopt the notation T_u and z' introduced in the proof of Lemma 3.7. We continue to let Z denote $k[u^2, v^2, x^2, y^2]$ and use $Z(R)$ to denote the center of R. We will prove that $Z = Z(R)$.

The algebra T_u is an Azumaya algebra, and the center of $T_u[z'; \sigma_u]$ is $Z[(z')^2]$, so it follows that the ramification locus of $T_u[z'; \sigma_u]$ is given by the prime ideals in $Z[(z')^2]$ that contain no elements of Z. Thus the ramification locus of $T_u[z'; \sigma_u]$ is $\langle (z')^2 \rangle$. Under the homomorphism $S_u[z; \sigma_u, \delta_u] \to R$, which sends z to u, the image of z' is nonzero in $R[\det(M_u)^{-1}]$; thus, passing from $T_u[z'; \sigma]$ to the quotient $R[\det(M_u)^{-1}]$, we see that the p.i.-degree of $R[\det(M_u)^{-1}]$ is the p.i.-degree of $T_u[z'; \sigma]$, which is four. Thus the algebra R also has p.i.-degree four over its center.

It remains to determine the center of R. Since R is a free module of rank sixteen over Z and since R has p.i.-degree four, its ring of quotients $Q(R)$ is a central, simple algebra over the ring of quotients, $Q(Z)$, of Z. Hence, if $Z \neq Z(R)$, then the ring of quotients, $Q(Z(R))$, of $Z(R)$ is $Q(R)$. In this case, there is an integral extension $Z \hookrightarrow Z(R)$, where Z is integrally closed, and Z and $Z(R)$ are birational. It follows that $Z = Z(R)$. ■

- **Case (b)** The map $S_v[z; \sigma_v, \delta_v] \to R$.

The subalgebra $S_v = Z[u, x, y]$ of R with defining relations

$$ux - xu = 0, \qquad uy - yu = 0, \qquad xy + yx = u^2 + v^2,$$

is the Clifford algebra $\mathrm{Cl}(Z[u], M_v)$ over $Z[u]$ associated to the symmetric matrix

$$M_v = \begin{pmatrix} 2x^2 & u^2 + v^2 \\ u^2 + v^2 & 2y^2 \end{pmatrix},$$

where, in the notation of Definition 3.2, $x_1 = x$ and $x_2 = y$. The determinant of M_v is $\det(M_v) = -(xy - yx)^2$. The following lemma follows from Proposition 3.5 and techniques similar to those of Lemma 3.6.

Lemma 3.9. *The subalgebra S_v of R is a domain of p.i.-degree two over its center $Z[u]$, which is a quadratic extension of Z.* ■

The Ore extension $S_v[z; \sigma_v, \delta_v]$, where $\sigma_v(u, v, x, y) = (-u, v, -x, -y)$ and $\delta_v(u, v, x, y) = (\lambda(xy - yx), 0, 0, 0)$, is well defined. As in the proof of Lemma 3.7, one may consider the localization T_v of S_v at the central element u. If we take $h = \delta_v(u)(\sigma_v(u) - u)^{-1} = -\frac{\lambda}{2}(xy - yx)u^{-1}$, then $T_v[z; \sigma_v, \delta_v] = T_v[z'; \sigma_v]$ where $z' = z + h$. The following result follows from mimicking the argument given for $S_u[z; \sigma_u, \delta_u]$.

Lemma 3.10.

The Ore extension $S_v[z; \sigma_v, \delta_v]$ is a domain of p.i.-degree four. ■

The homomorphism $S_v[z; \sigma_v, \delta_v] \to R$ determined by sending z to v may be used to give another proof of Theorem 3.8.

• **Case (c)** The map $S_x[z; \sigma_x, \delta_x] \to R$.

Recall that the normal element $\Omega = xy - yx$ of R commutes with both u and v. The algebra $U_x = Z[xy - yx, u, v]$ with defining equation

$$uv + vu = \lambda(xy - yx)$$

is the Clifford algebra $\mathrm{Cl}(Z[xy - yx], M_x)$ over $Z[xy - yx]$ associated to

$$M_x = \begin{pmatrix} 2u^2 & \lambda(xy - yx) \\ \lambda(xy - yx) & 2v^2 \end{pmatrix},$$

which has nonzero determinant $\det(M_x) = -(uv - vu)^2 \in Z$. As in Definition 3.2, we have $x_1 = u, x_2 = v$. By Proposition 3.5, we have that U_x is a domain of p.i.-degree two over its center $Z[xy - yx]$.

The linear map σ' which sends $(u, v, xy - yx)$ to $(u, -v, -(xy - yx))$ but is trivial on Z, extends to an automorphism of the algebra U_x and so we may define the skew-polynomial algebra $U_x[Y; \sigma']$ and an algebra homomorphism $U_x[Y; \sigma'] \mapsto R$ by sending Y to y. The image of this homomorphism is a subalgebra of R which we call S_x.

Lemma 3.11. *The algebras $U_x[Y; \sigma']$ and S_x are domains of p.i.-degree four with center Z.*

Proof. The result for $U_x[Y; \sigma']$ follows from the fact that σ' acts nontrivially on the center of U_x. The remainder of the proof follows from arguments analogous to those given above; that is, localizing U_x at the normal element $uv - vu$ and noting that the ramification divisor of the skew-polynomial ring $U_x[(uv - vu)^{-1}][Y; \sigma']$ is determined by $\langle Y \rangle$ and so S_x has the same p.i.-degree as $U_x[Y; \sigma']$. ■

The linear map σ_x defined by $\sigma_x(u, v, x, y) = (u, -v, -x, -y)$ extends to an automorphism of S_x and the linear map δ_x defined by $\delta_x(u, v, x, y) = (0, 0, 0, u^2 + v^2)$ extends to a σ_x-derivation on S_x, so we may define the Ore extension $S_x[z; \sigma_x, \delta_x]$. Moreover, there is a homomorphism $S_x[z; \sigma_x, \delta_x] \to R$ which is defined by sending z to $x \in R$.

Lemma 3.12. *The Ore extension $S_x[z;\sigma_x,\delta_x]$ is a domain of p.i.-degree four.*

Proof. The automorphism σ_x acts trivially on the center Z of the p.i.-domain S_x, and so, by the Skolem-Noether Theorem, [6, §10.7], σ_x is inner on some localization of S_x. Indeed, the automorphism σ_x may be obtained via conjugation of the normal element $y(xy-yx)$ of S_x. If T_x is the localization obtained from S_x by inverting the normal element $y(xy-yx)$, then we may write

$$T_x[z;\sigma_x,\delta_x] = T_x[z';\delta'],$$

where $z' = y(xy-yx)z$ and $\delta'(a) = y(xy-yx)\delta_x(a)$ for all $a \in T_x$. However, δ' is trivial on the center of S_x, so again, by the Skolem-Noether Theorem, δ' is an inner derivation. In fact, $\delta'(a) = [a, xy(xy-yx)]$. Finally, replacing z' by $z'' = z' + xy(xy-yx)$ yields

$$T_x[z;\sigma_x,\delta_x] = T_x[z''] = T_x[y(xy-yx)z + xy(xy-yx)],$$

which proves that $T_x[z;\sigma_x,\delta_x]$ is a domain of p.i.-degree four. ■

The homomorphism $S_x[z;\sigma_x,\delta_x] \to R$ may be used to give a third proof of Theorem 3.8.

- **Case (d)** The map $S_y[z;\sigma_y,\delta_y] \to R$.

The fourth realization of R as the quotient of an Ore extension is analogous to the preceding work. If U_y denotes the subalgebra of R generated over the commutative subring $Z[xy-yx]$ by u and v, then it is the Clifford algebra $\mathrm{Cl}(Z[xy-yx], M_y)$, where

$$M_y = \begin{pmatrix} 2u^2 & \lambda(xy-yx) \\ \lambda(xy-yx) & 2v^2 \end{pmatrix}$$

has nonzero determinant $\det(M_y) = -(uv-vu)^2$. The Ore extension $U_y[X;\sigma']$, where σ' is trivial on Z but maps $(u,v,xy-yx)$ to $(u,-v,-(xy-yx))$, is well defined. We may form the skew-polynomial algebra $U_y[X;\sigma']$ and an algebra map $U_y[X;\sigma'] \to R$, which sends X to x. Let the image of this map be S_y. If $\sigma_y(u,v,x,y) = (u,-v,-x,-y)$ and if $\delta_y(u,v,x,y) = (0,0,u^2+v^2,0)$, then we have a homomorphism $S_y[z;\sigma_y,\delta_y] \to R$ by sending z to y. The ramification divisor of S_y is determined by the central element $-(uv - vu)^2$. The Ore extension $S_y[z; \sigma_y, \delta_y]$ is a domain of p.i.-degree four and the same holds for R.

4. The Central Proj of R

Since R is a finite module over its center $Z = k[u^2, v^2, x^2, y^2]$, we may associate to R a coherent sheaf of algebras $\mathcal{R}$ over $\operatorname{Proj} Z \cong \mathbb{P}^3$, c.f., [7, Chapter II, §2]. The sheaf $\mathcal{R}$ is defined by its local section algebras $\Gamma_z = (R[z^{-2}])_0$ over each of the four affine pieces $\operatorname{Spec} Z[z^{-2}]_0 \cong \mathbb{A}^3$, with $z = u, v, x, y$ in turn. A description of these section algebras follows in Section 4.1.

The center $\mathcal{Z}$ of the sheaf $\mathcal{R}$ is a coherent sheaf of commutative algebras over $\operatorname{Proj} Z$, defined by $\mathcal{Z}(U) = Z(\mathcal{R}(U))$ for an open set $U \subset \operatorname{Proj} Z$, where $Z(\mathcal{R}(U))$ denotes the center of the algebra $\mathcal{R}(U)$. To the sheaf $\mathcal{Z}$, we may associate a commutative scheme $X = \mathbf{Spec}\mathcal{Z}$ together with an affine morphism

$$f : X = \mathbf{Spec}\mathcal{Z} \longrightarrow Y = \operatorname{Proj} Z = \mathbb{P}^3$$

such that $f_*\mathcal{O}_X \cong \mathcal{Z}$ and such that f_* induces an equivalence between $\mathcal{O}_Y$-modules and $\mathcal{Z}$-modules (see [7, Ex. II.5.17]). The scheme $\mathbf{Spec}\mathcal{Z}$ is obtained by gluing the affine schemes $\operatorname{Spec} Z(\Gamma_z)$, where $Z(\Gamma_z)$ denotes the center of the section algebra Γ_z and $z = u, v, x, y$ respectively. We call the scheme, $\mathbf{Spec}\mathcal{Z}$, the central Proj of R. Note that the central Proj of R need not coincide with the Proj of the center of R.

Example 4.1. In [2], it is shown that in the case of the three-dimensional Sklyanin algebras, $\mathbf{Spec}\mathcal{Z} \cong \mathbb{P}^2$. An alternative proof is given in [18], where the authors also describe the central Proj of the four-dimensional Sklyanin algebra. In the latter case, $\mathbf{Spec}\mathcal{Z} \ncong \mathbb{P}^3$ because $\mathbf{Spec}\mathcal{Z}$ contains singularities.

The following result is proved in Theorems 4.10 and 4.12.

> The sheaf of algebras $\mathcal{R}$ over $\operatorname{Proj} Z$ is a sheaf of quaternion algebras over its center $\mathcal{Z}$. The central Proj, $\mathbf{Spec}\mathcal{Z}$, may be

identified with a smooth quadric in $\mathbb{P}^4$, and thus it determines a degree-two cover of $\operatorname{Proj} Z$.

The first part of this statement, concerning the p.i.-degree of the sheaf $\mathcal{R}$, may be deduced from Theorem 3.8 and [9, p. 149] using the fact that the p.i.-degree of R is $n = 4$ and that the greatest common divisor of the degrees of central homogeneous elements in R is $m = 2$, so that it follows that the p.i.-degree of $\mathcal{R}$ is $\frac{n}{m} = 2$. Nevertheless, we prefer to study the local section algebras Γ_z, where $z = u$, v, x, y respectively because this allows a more explicit description of the scheme $\mathbf{Spec}\mathcal{Z}$ in Section 4.2.

Let A be a positively graded, affine k-algebra which is connected and generated by degree-one elements such that A is a domain and a finite module over its center Z. The scheme $\operatorname{Proj} Z$ is covered by affine open subsets $\operatorname{Spec}(Z_z)_0$, where $z \in Z$ is homogeneous and $(Z_z)_0$ denotes the localization of Z at $\{1, z, z^2, \ldots\}$. The structure sheaf $\mathcal{A}$ of A over $\operatorname{Proj} Z$ is defined by its sections $\Gamma(\operatorname{Spec}(Z_z)_0, \mathcal{A}) = (A_z)_0$ for all homogeneous $z \in Z$.

In analogy with the ramification locus of A, we define the ramification locus of the structure sheaf $\mathcal{A}$ of A as follows. As in [9, Page 146], let e be the p.i.-degree of the sheaf of algebras $\mathcal{A}$, and let $\mathbf{Spec}\mathcal{Z}$ be the central Proj of A. The *Azumaya locus* of $\mathcal{A}$ is the open subset of $\mathbf{Spec}\mathcal{Z}$ consisting of the points corresponding to the simple e-dimensional representations of $\mathcal{A}$. The complement of the Azumaya locus in $\mathbf{Spec}\mathcal{Z}$ is called the *ramification locus* of $\mathcal{A}$, which thus corresponds to the e-dimensional semisimple representations of $\mathcal{A}$ that are not simple.

Suppose, in addition, that A has finite injective dimension. If f is the affine morphism $f : \mathbf{Spec}\mathcal{Z} \to \mathbb{P}^3$ and if $\mathcal{A}$ is the structure sheaf over $\mathbb{P}^3$ associated to A, then we have the following result.

[9, Theorem 1] If the ramification locus of $f^*\mathcal{A}$ is not pure of codimension one, then the central Proj, $\mathbf{Spec}\mathcal{Z}$, is singular.

Proposition 4.14 of the sequel shows that, in our setting, the ramification locus of $\mathcal{R}$ is pure of codimension one; therefore, one cannot apply the result of [9, Theorem 1] to obtain singularities in the central Proj of R. On the contrary, Theorem 4.12 shows that the central Proj of R is a nonsingular quadric in $\mathbb{P}^4$, and thus the above result of [9, Theorem 1] cannot be improved.

4.1. The section algebras $\Gamma_z = R[z^{-2}]_0$.

- **Case (a)** The section algebra Γ_u.

In order to describe $\Gamma_u = R[u^{-2}]_0$, the surjective map $S_v[z;\sigma_v,\delta_v] \to R$ of Section 3 Case (b) may be used. We first consider $\Sigma_u = S_v[u^{-1}]_0$, which is generated by $X_u = xu^{-1}, Y_u = yu^{-1}$ and the central element $H_u = v^2u^{-2}$.

Lemma 4.2. *The algebra Σ_u is a quaternion algebra with center $k[X_u^2, Y_u^2, H_u]$ and ramification locus determined by $-(X_uY_u - Y_uX_u)^2 = -(xy-yx)^2u^{-4}$.*

Proof. Clearly,

$$\begin{aligned} X_uY_u &= xu^{-1}yu^{-1} \\ &= xyu^{-2} \\ &= (-yx+u^2+v^2)u^{-2} \\ &= -Y_uX_u + 1 + H_u, \end{aligned}$$

hence Σ_u is a Clifford algebra over $k[X_u^2, Y_u^2, H_u]$ with defining associated symmetric matrix

$$\begin{pmatrix} 2X_u^2 & 1+H_u \\ 1+H_u & 2Y_u^2 \end{pmatrix},$$

from which the result follows. ■

Before we describe Γ_u as a quotient of an Ore extension over Σ_u, we need the commutation relations of $V_u = vu^{-1}$ with X_u and Y_u, which are

$$\begin{aligned} V_uX_u - X_uV_u &= -\lambda X_u(X_uY_u - Y_uX_u), \text{ and} \\ V_uY_u - Y_uV_u &= -\lambda Y_u(X_uY_u - Y_uX_u). \end{aligned}$$

It follows that the Ore extension $\Sigma_u[z;\Delta_u]$, where Δ_u is the inner derivation of Σ_u given by

$$\Delta_u(a) = [a, -\lambda X_uY_u],$$

is well defined. By sending z to V_u we obtain a surjective homomorphism $\Sigma_u[z;\Delta_u] \to \Gamma_u$. Recall from the techniques used in Section 3 that we may write $\Sigma_u[z;\Delta_u] = \Sigma_u[z']$ if one writes $z' = z - \lambda X_u Y_u$.

Lemma 4.3.

(a) *The Ore extension* $\Sigma_u[z;\Delta_u] = \Sigma_u[z']$ *is a quaternion algebra with center* $k[H_u, X_u^2, Y_u^2, z']$.

(b) *The section algebra* Γ_u *is a quaternion algebra. The center of* Γ_u *is* $k[H_u, X_u^2, Y_u^2, V_u - \lambda X_u Y_u]$, *which is a quadratic extension of the section algebra* $k[H_u, X_u^2, Y_u^2]$ *of* Proj Z. *The central* Proj *of* R *is smooth in the open set determined by* u^2.

Proof. The minimal polynomial of $V_u - \lambda X_u Y_u$ over $k[H_u, X_u^2, Y_u^2]$ is

$$(V_u - \lambda X_u Y_u)^2 + \lambda(X_u Y_u + Y_u X_u)(V_u - \lambda X_u Y_u) + \lambda^2 X_u^2 Y_u^2 + H_u,$$

where $X_u Y_u + Y_u X_u = 1 + H_u$, so it has degree two. Thus, the sections of $f_* \mathcal{O}_X \cong \mathcal{Z}$ over the open set determined by u^2 are given by the affine commutative algebra

$$\frac{k[H_u, X_u^2, Y_u^2, z']}{\langle (z')^2 + \lambda(1 + H_u)z' + \lambda^2 X_u^2 Y_u^2 + H_u \rangle},$$

and the central Proj is smooth in this open set since $\lambda^2 \neq 1$ (c.f., Section 1). ■

• **Case (b)** The section algebra Γ_v.

We use the description $S_u[z;\sigma_u,\delta_u] \to R$ of Section 3 Case (a) to study $\Sigma_v = (S_u[v^{-1}])_0$, which is generated by the elements $X_v = xv^{-1}$, $Y_v = yv^{-1}$ and the central element $H_v = u^2 v^{-2}$. For the gluing procedure in Theorem 4.10, it is useful to note that $X_v^2 = -x^2 v^{-2}$ and $Y_v^2 = -y^2 v^{-2}$.

Lemma 4.4. *The algebra* Σ_v *is a quaternion algebra with center* $k[X_v^2, Y_v^2, H_v]$ *and with ramification locus determined by* $-(X_v Y_v - Y_v X_v)^2 = -(xy - yx)^2 v^{-4}$.

Proof. A computation shows that $X_vY_v + Y_vX_v = -(1 + H_v)$, hence Σ_v is the Clifford algebra over $k[X_v^2, Y_v^2, H_v]$ associated to the symmetric matrix

$$\begin{pmatrix} 2X_v^2 & -(1+H_v) \\ -(1+H_v) & 2Y_v^2 \end{pmatrix},$$

from which the result follows. ■

The commutation relations of X_v and Y_v with $U_v = uv^{-1}$ are

$$\begin{aligned} U_vX_v - X_vU_v &= -\lambda X_v(Y_vX_v - X_vY_v), \text{ and} \\ U_vY_v - Y_vU_v &= -\lambda Y_v(Y_vX_v - X_vY_v). \end{aligned}$$

To study $\Gamma_v = R[v^{-2}]_0$, we first define the Ore extension $\Sigma_v[z; \Delta_v]$, where Δ_v is the inner derivation of Σ_v determined by

$$\Delta_v(a) = [a, -\lambda Y_vX_v].$$

Sending z to U_v yields a surjective map $\Sigma_v[z; \Delta_v] \to \Gamma_v$. As in Section 4 Case (a), we may replace z by $z' = z - \lambda Y_vX_v$ to obtain the following result.

Lemma 4.5.

(a) *The Ore extension* $\Sigma_v[z; \Delta_v] = \Sigma_v[z']$ *is a quaternion algebra with center* $k[H_v, X_v^2, Y_v^2, z']$.

(b) *The section algebra* Γ_v *is a quaternion algebra. The center of* Γ_v *is* $k[H_v, X_v^2, Y_v^2, U_v - \lambda Y_vX_v]$*, which is a quadratic extension of the section algebra* $k[H_v, X_v^2, Y_v^2]$ *of* Proj Z*. The central* Proj *of* $\mathcal{R}$ *is smooth in the open set determined by* v^2*.*

Proof. As in Lemma 4.3, the most cumbersome part of the proof is to find the minimal polynomial of $U_v - \lambda Y_vX_v$ over the section algebra $k[H_v, X_v^2, Y_v^2]$ of Proj Z. A calculation yields

$$(U_v - \lambda Y_vX_v)^2 + \lambda(X_vY_v + Y_vX_v)(U_v - \lambda Y_vX_v) + \lambda^2 X_v^2 Y_v^2 + H_v = 0,$$

where $X_vY_v + Y_vX_v = -(1 + H_v)$. It follows that the central Proj is smooth in the affine open set determined by v^2. ■

• **Case (c)** The section algebra Γ_x.

Although we may proceed here as above by starting with a Clifford subalgebra, we instead choose a different approach. The generators $U_x = ux^{-1}$, $V_x = vx^{-1}$ and $Y_x = yx^{-1}$ of $\Gamma_x = R[x^{-2}]_0$ satisfy the following commutation relations

$$\begin{aligned} V_xU_x - U_xV_x &= -2\lambda Y_x + \lambda(U_x^2 - V_x^2), \\ Y_xU_x + U_xY_x &= (U_x^2 - V_x^2)U_x, \qquad \text{and} \\ Y_xV_x + V_xY_x &= (U_x^2 - V_x^2)V_x. \end{aligned}$$

Thus the Ore extension

$$k[U_x^2, V_x^2, Y_x][z_1; \sigma_1]$$

is well defined, where $\sigma_1(U_x^2, V_x^2, Y_x) = (U_x^2, V_x^2, -Y_x + U_x^2 - V_x^2)$, and we let Σ_x be the image in Γ_x under the map $k[U_x^2, V_x^2, Y_x][z_1; \sigma_1] \to \Gamma_x$ which sends z_1 to U_x. Let H_x denote y^2x^{-2}.

Lemma 4.6. *The algebra Σ_x is a quaternion algebra with center $k[H_x, U_x^2, V_x^2]$ and with ramification locus determined by $-(U_xV_x - V_xU_x)^2 = -\lambda^2(xy - yx)^2x^{-4}$.*

Proof. Since σ_1 has order two, the center of $k[U_x^2, V_x^2, Y_x][z_1; \sigma_1]$ is the algebra $k[U_x^2, V_x^2, Y_x(Y_x - U_x^2 + V_x^2)][z_1^2]$, where $-Y_x(Y_x - U_x^2 + V_x^2) = H_x$. The claims now follow as in the preceding cases. ■

The linear maps

$$\begin{aligned} \sigma_2(V_x^2, Y_x, U_x) &= (V_x^2, -Y_x + U_x^2 - V_x^2, U_x) \qquad \text{and} \\ \delta_2(V_x^2, Y_x, U_x) &= (0, 0, -2\lambda Y_x + \lambda(U_x^2 - V_x^2)) \end{aligned}$$

extend to an automorphism of Σ_x and a σ_2-derivation of Σ_x respectively. Hence, the section algebra Γ_x is obtained as the image of the map $\Sigma_x[z_2; \sigma_2, \delta_2] \to \Gamma_x$ which sends z_2 to V_x.

Lemma 4.7. *The section algebra Γ_x is a quaternion algebra. The center of Γ_x is the algebra $k[H_x, U_x^2, V_x^2, U_xV_x + V_xU_x]$, which is a quadratic extension of the section algebra $k[H_x, U_x^2, V_x^2]$ of* Proj Z. *The central* Proj *of R is smooth in the open set determined by x^2.*

Proof. We first concentrate on the Ore extension $\Sigma_x[z_2;\sigma_2,\delta_2]$, and let Σ be the localization of Σ_x at U_x. As σ_2 is the inner automorphism of Σ_x obtained by conjugation with the normal element U_x, it follows that

$$\Sigma[z_2;\sigma_2,\delta_2]=\Sigma[z_2;\delta_2'],$$

where $\delta_2'(a)=U_x\delta_2(a)$ is the inner derivation on Σ_x determined by $\delta_2'(a)=[a,V_xU_x]$. Hence, the Ore extension $\Sigma[z_2;\sigma_2,\delta_2]$ is a quaternion algebra with center $k[U_x^2,V_x^2,H_x,U_xV_x+V_xU_x]$, from which follow the claims about Γ_x.

It remains to find the degree of the extension. However, the element $U_xV_x+V_xU_x$ is central in Γ_x and

$$(U_xV_x+V_xU_x)^2+(2\lambda^2-4)U_x^2V_x^2-\lambda^2(U_x^4+V_x^4)+4\lambda^2H_x=0,$$

so the sections of the central Proj are smooth in the open subset determined by x^2. ■

The above suggests that it might be possible to describe Γ_x by using the Clifford algebra over $k[U_x^2,V_x^2,U_xV_x+V_xU_x]$ corresponding to the symmetric matrix

$$\begin{pmatrix} 2U_x^2 & U_xV_x+V_xU_x \\ U_xV_x+V_xU_x & 2V_x^2 \end{pmatrix}.$$

However, another interpretation of the section algebra Γ_x is as a graded Artin-Schelter regular algebra of global dimension three generated by two elements U_x and V_x which satisfy the cubic relations

$$U_xV_x^2=V_x^2U_x \quad \text{and} \quad V_xU_x^2=U_x^2V_x$$

as in [4]. Owing to [9, Lemma 1], Γ_x is Auslander-regular and is of type S_1 (c.f., [3]) with point scheme $\mathbb{P}^1\times\mathbb{P}^1$.

- **Case (d)** The section algebra Γ_y.

Once more, this analysis will be analogous to the preceding cases. In summary, if $U_y=uy^{-1}$, $V_y=vy^{-1}$ and $X_y=xy^{-1}$ are the generators of

$\Gamma_y = R[y^{-2}]_0$, then

$$\begin{aligned} V_yU_y - U_yV_y &= 2\lambda X_y - \lambda(U_y^2 - V_y^2), \\ X_yU_y + U_yX_y &= (U_y^2 - V_y^2)U_y, \quad \text{and} \\ X_yV_y + V_yX_y &= (U_y^2 - V_y^2)V_y. \end{aligned}$$

Defining $\sigma_1(U_y^2, V_y^2, X_y) = (U_y^2, V_y^2, -X_y + U_y^2 - V_y^2)$, we let Σ_y be the image of the Ore extension $k[U_y^2, V_y^2, X_y][z_1; \sigma_1]$ in Γ_y under the map obtained by sending $z_1 \mapsto U_y$. Let $H_y = x^2y^{-2}$.

Lemma 4.8. *The algebra Σ_y is a quaternion algebra with center $k[H_y, U_y, V_y^2]$ and with ramification locus determined by*

$$-(U_yV_y - V_yU_y)^2 = -\lambda^2(xy - yx)^2y^{-4}. \qquad \blacksquare$$

The linear maps

$$\begin{aligned} \sigma_2(V_y^2, X_y, U_y) &= (V_y^2, -X_y + U_y^2 - V_y^2, U_y) \quad \text{and} \\ \delta_2(V_y^2, X_y, U_y) &= (0, 0, 2\lambda X_y - \lambda(U_y^2 - V_y^2)) \end{aligned}$$

extend to an automorphism of Σ_y and a σ_2-derivation of Σ_y respectively. Hence, the section algebra Γ_y is obtained as the image of the map $\Sigma_y[z_2; \sigma_2, \delta_2] \to \Gamma_y$ which sends z_2 to V_y.

Lemma 4.9. *The section algebra Γ_y is a quaternion algebra. The center of Γ_y is the algebra $k[H_y, U_y^2, V_y^2, U_yV_y + V_yU_y]$, which is a quadratic extension of the section algebra $k[H_y, U_y^2, V_y^2]$ of $\mathbb{P}^3$. The central* Proj *of R is smooth in the open set determined by y^2.*

Proof. The proof is similar to that in the earlier cases. We find that the element $U_yV_y + V_yU_y$ is central in Γ_y and that

$$(U_yV_y + V_yU_y)^2 + (2\lambda^2 - 4)U_y^2V_y^2 - \lambda^2U_y^4 - \lambda^2V_y^4 + 4\lambda^2H_y = 0. \qquad \blacksquare$$

These results bring us to one of our main theorems.

Theorem 4.10. *The sheaf $\mathcal{R}$ is a sheaf of quaternion algebras over its center $\mathcal{Z}$. The central* Proj, **Spec**$\mathcal{Z}$, *is smooth and corresponds to a degree-two cover of $\mathbb{P}^3 = \operatorname{Proj} Z$.*

Proof. This result follows from Lemmas 4.3, 4.5, 4.7 and 4.9 as follows. The local section algebras $\Gamma_z = R[z^{-2}]_0$, where $z = u, v, x, y$ respectively, are quaternion algebras over their centers $Z(R[z^{-2}]_0)$, and the local section algebras $Z(R[z^{-2}]_0)$ of the sheaf $\mathcal{Z}$ are quadratic extensions of the section algebras $(Z[z^{-2}])_0$ of $\mathbb{P}^3 = \operatorname{Proj} Z$. The central Proj of R was shown to be smooth in each of the open sets determined by z^2. ■

4.2. **The central** Proj**, a smooth quadric in** $\mathbb{P}^4$**.**

In Theorem 4.10, we showed that the scheme $\mathbf{Spec}\mathcal{Z}$ is the scheme associated to a smooth projective variety that corresponds to a degree-two cover of $\mathbb{P}^3 = \operatorname{Proj} Z$. In this subsection, we identify this projective variety and describe its relationship to the algebra R.

Since R has p.i.-degree four over its center Z, a generic element of R satisfies a degree-four polynomial over Z. However, certain elements might be quadratic over Z, such as the element $uv - \lambda xy$, whose minimal polynomial over Z is

$$X^2 + \lambda(u^2 + v^2)X + \lambda^2 x^2 y^2 + u^2 v^2 \in Z[X].$$

Recall from Section 1 that $\lambda^2 \neq 0, 1$.

Lemma 4.11. *The equation*

$$X^2 + \lambda(u^2 + v^2)X + \lambda^2 x^2 y^2 + u^2 v^2 = 0$$

determined by the minimal polynomial of $uv - \lambda xy$*, where* $\lambda \in k^\times \setminus \{\pm 1\}$*, determines a smooth quadric* Q_λ *in* $\mathbb{P}^4 = \operatorname{Proj}(k[u^2, v^2, x^2, y^2, X])$. ■

Theorem 4.12. *The central* Proj*,* $\mathbf{Spec}\mathcal{Z}$*, of* R *is the smooth quadric* Q_λ *in* $\mathbb{P}^4$ *which is determined by the minimal polynomial of the element* $uv - \lambda xy \in R$ *over the center* Z *of* R*.*

Proof. It suffices to check that the local section algebras of the scheme defined by the quadric Q_λ are the section algebras Γ_z, where $z = u, v, x, y$ respectively. However, the above implies that the global section algebra of the scheme defined by the quadric Q_λ, is the algebra $k[u^2, v^2, x^2, y^2, uv - \lambda xy]$ which is $k[u^2, v^2, x^2, y^2, X]/\langle X^2 + \lambda(u^2+v^2)X + \lambda^2x^2y^2 + u^2v^2\rangle$.

Given that $Y = \operatorname{Proj} Z = \mathbb{P}^3$, it follows that the local section algebra of the scheme Q_λ over the basic open set $Y(u^2)$ is $k[H_u, X_u^2, Y_u^2, V_u - \lambda X_u Y_u]$, where $H_u = v^2u^{-2}$, $X_u = xu^{-1}$ and $Y_u = yu^{-1}$. This algebra is the section algebra Γ_u as described in the proof of Lemma 4.3.

Moreover, the same holds for the other basic open sets. However, on the basic open set $Y(v^2)$, it is better to use the representation $uv - \lambda xy$, and on $Y(x^2)$ and $Y(y^2)$ one should use the minimal polynomial of $uv - vu$, which is $W^2 + 4\lambda^2x^2y^2 - (2\lambda^2 - 4)u^2v^2 - \lambda^2(u^4 + v^4) \in Z[W]$. This polynomial defines the same quadric Q_λ in $\mathbb{P}^4 = \operatorname{Proj} k[u^2, v^2, x^2, y^2, W]$ which is isomorphic to $\operatorname{Proj} k[u^2, v^2, x^2, y^2, X]$ since $W = 2X + \lambda(u^2 + v^2)$. Therefore, $Z[uv - \lambda xy] = Z[uv - vu]$. ■

It follows that the degree-two cover of the central Proj over $\mathbb{P}^3$ corresponds to the inclusion $Z \subset Z[uv - vu]$.

One may also give an interpretation of the quadric Q_λ in terms of the 2-Veronese $R^{(2)}$ of R, defined by $R^{(2)} = \bigoplus_{n\in\mathbb{N}} R_{2n}$. By [18, Lemma 5.1], it follows that $\mathbf{Spec}\mathcal{Z} \cong \operatorname{Proj} Z(R^{(2)})$, so that the following result is immediate.

Proposition 4.13. *The center of the* 2*-Veronese of* R *is* $Z[uv - vu]$*, which is a quadratic extension of* Z*.* ■

We return to the study of the ramification locus and Azumaya locus of the sheaf $\mathcal{R}$ because they are related to the study of fat point modules. More details may be found in [9, §2], from which it follows that the study of the fat point modules of multiplicity two reduces to the study of the Azumaya locus $U \subset \mathbf{Spec}\mathcal{Z}$ (introduced at the start of Section 4). Since the p.i.-degree of the sheaf $\mathcal{R}$ is two, there are no fat point modules of multiplicity higher than two, whereas the fat point modules of multiplicity one are simply the point modules.

Thus, our interest lies in the complement of U in $\mathbf{Spec}\mathcal{Z}$, which was called the ramification locus.

Proposition 4.14. *The ramification locus of* $\mathcal{R}$ *is* $\mathcal{V}((xy - yx)^2) \cap Q_\lambda$.

Proof. This result follows from gluing the ramification loci of the section algebras Γ_z, where $z = u,\ v,\ x,\ y$ respectively; that is, gluing the ramification loci $\mathcal{V}((xy - yx)^2 z^{-4})$ of the corresponding algebra Σ_z, where $z = u, v, x$ or y and noting that $(xy - yx)^2 = v^4 - 4x^2y^2 + 2u^2v^2 + u^4 \in Z$. ■

At this stage, three quadrics have been introduced: the point scheme $Q \subset \mathbb{P}^3$, the central Proj, $Q_\lambda \subset \mathbb{P}^4$, and $\mathcal{V}((xy - yx)^2)$. Consider the projection map

$$\Pi : Q_\lambda \subset \operatorname{Proj} k[u^2, v^2, x^2, y^2, X] = \mathbb{P}^4 \to \mathbb{P}^3 = \operatorname{Proj} k[u^2, v^2, x^2, y^2].$$

This projection map defines a ramification locus in $\Pi(Q_\lambda)$, which is the locus of the points p in $\Pi(Q_\lambda)$ for which $\Pi^{-1}(p)$ contains only one point of Q_λ instead of two. Recall from Theorem 4.12 that $Q_\lambda = \mathcal{V}(X^2 + aX + b) = \mathcal{V}(W^2 + d)$ for certain $a, b, d \in k[u^2, v^2, x^2, y^2]$. Thus, the ramification locus C_λ of the above projection is the quadric $C_\lambda = \mathcal{V}(a^2 - 4b) = \mathcal{V}(d)$ in $\mathbb{P}^3$, that is

$$C_\lambda = \mathcal{V}((2\lambda^2 - 4)u^2v^2 + \lambda^2(u^4 + v^4) - 4\lambda^2x^2y^2) \subset \mathbb{P}^3.$$

The projection Π maps the lines on Q_λ to lines that either lie on C_λ or are tangential to C_λ.

It would be satisfying to find a connection between the quadrics Q, Q_λ, C_λ and $\mathcal{V}((xy - yx)^2)$ associated to R and the non-commutative projective geometry of R, such as (fat) point and line modules.

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