SOME FINITE QUANTUM $\mathbb{P}^3$S THAT ARE INFINITE MODULES OVER THEIR CENTERS

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Abstract. A result of M. Artin, J. Tate and M. Van den Bergh asserts that a regular algebra of global dimension three is a finite module over its center if and only if the automorphism encoded in the point scheme has finite order. We prove that the analogous result for a regular algebra of global dimension four is false by presenting families of quadratic, noetherian regular algebras A of global dimension four such that (i) A is an infinite module over its center, (ii) A has a finite point scheme, which is the graph of an automorphism of finite order, and (iii) A has a one-parameter family of line modules. Such algebras are candidates for generic regular algebras of global dimension four. The methods used to construct the algebras provide new techniques for creating other potential candidates.

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Introduction

Artin-Schelter regular algebras often share many homological and algebraic properties in common with polynomial algebras, so regular algebras are often viewed as non-commutative analogues of polynomial algebras. In this spirit, associated to a regular algebra is a geometry of modules, and, in many cases, this geometry depicts the algebraic behavior of the algebra. Using the geometric language from [1], a regular algebra of global dimension three is called a quantum $\mathbb{P}^2$. A quadratic, regular algebra of global dimension four that satisfies certain additional homological conditions (such as any of the algebras presented in Section 2) is sometimes called a quantum $\mathbb{P}^3$.

If A is a connected, regular algebra of global dimension three that is generated by degree-one elements, then the associated geometry entails, in part, a scheme (the *point scheme*) that parametrizes certain graded modules (*point modules*) over A. In this setting, the point scheme is the graph of an automorphism τ_A of a subscheme of $\mathbb{P}(A_1^*)$ ([2]). Under a few extra homological conditions, similar results concerning point modules (and *line modules*) hold for connected, quadratic regular algebras of global dimension four ([7, Theorem 1.4], [9, Theorem 1.10]).

In the late 1980s, M. Artin, J. Tate and M. Van den Bergh proved that, in the global-dimension-three setting, a regular algebra A that is generated by degree-one elements is a finite module over its center if and only if the automorphism τ_A has finite order ([3, Theorem II]). Until now, it has been unknown if such a result holds in the global-dimension-four case, even if additional homological conditions are assumed. We address this issue in our main result as follows.

Theorem. *Let k denote a field of characteristic zero. There exist quadratic, Artin-Schelter regular k-algebras A of global dimension four such that*

(a) *A is an infinite module over its center,*
(b) *the point scheme of A is finite,*
(c) *the automorphism τ_A has finite order,*
(d) *A has a one-parameter family of line modules.*

Families of algebras that satisfy this theorem are given in Propositions 2.1 and 2.2.

It seems reasonable to conjecture that a theorem analogous to Theorem II in [3] should hold in the global-dimension-four setting, but, in light of our main theorem, perhaps the analogue should use instead the line scheme (defined in [8]) and some associated geometric data, in place of the point scheme and τ_A.

An unsolved problem is the classification of regular algebras of global dimension four; in fact, even the *generic classes* of regular algebras of global dimension four are unknown. A generic quadratic algebra on four generators with six defining relations has a finite point scheme, and such an algebra that is regular has a one-parameter family of line modules. Since only two such algebras were previously known ([6, 10]), the families of algebras presented in this article provide additional welcome candidates, as they give new examples on which conjectures pertaining to generic regular algebras may be tested.

If one views the family of all quadratic algebras with four generators and six relations as a scheme $\mathcal{S}$, then it is unclear how the algebras presented herein compare to the algebras presented in [6] that have twenty distinct points and a one-parameter family of line modules. Conceivably, the algebras in [6] lie in a different component of $\mathcal{S}$ than the algebras of our main theorem; on the other hand, if they lie in the same component, then it is reasonable to assume there exists an algebra with similar properties to those of our main result, but which additionally has twenty distinct points.

This article is outlined as follows. In Section 1, we define the terminology to be used in the paper, and give a simple criterion that implies that certain types of Ore extensions are infinite modules over their centers. In Section 2, we give two classes of regular algebras of global dimension four to which the criterion applies and which prove our main theorem.

1. Preliminaries

In this section, k denotes a field of characteristic not equal to two. The algebras to be considered will be associative k-algebras A such that $A = \bigoplus_{i \geq 0} A_i$ as a vector space, $A_i A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{N} \cup \{0\}$, $\dim_k(A_i) < \infty$ for all i, and $A_0 = k$. We denote the one-dimensional graded A-bimodule $A/\langle A_1 \rangle$ by either ${}_A k$ or k_A depending on whether we are considering its left or right A-module structure. If $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a graded A-module such that $M_i = 0$ for all $i \ll 0$, then the *Hilbert series* of M is the Laurent power series $H_M(t) = \sum_i (\dim_k M_i) t^i$.

Definition. [2] A graded algebra A, as above, is *Artin-Schelter regular* (or simply, *regular*) of global dimension n if A has finite global dimension n and polynomial growth, and if A is

Gorenstein, meaning that

$$\mathrm{Ext}^i_A({}_Ak, A) = \begin{cases} 0 & \text{if } i \neq n \\ k_A \text{ (shifted)} & \text{if } i = n. \end{cases}$$

If A is a quadratic algebra, then a *point module* (respectively, *line module*) over A is defined to be a graded, cyclic A-module M such that $H_M(t) = (1-t)^{-1}$ (respectively, $H_M(t) = (1-t)^{-2}$).

If A is a quadratic regular algebra of global dimension three, then there is a scheme (called the *point scheme* in [9]) that parametrizes the point modules; it may be computed as the zero locus, Γ_A, in $\mathbb{P}^2 \times \mathbb{P}^2$ of the defining relations of A ([2, Section 3]); here, $\mathbb{P}^2 = \mathbb{P}(A_1^*)$. By [2], [7, Theorem 1.4] and [9, Theorem 1.10], this result extends to the global dimension-four setting, if one additionally assumes that A is noetherian, Auslander-regular, satisfies the Cohen-Macaulay property and has Hilbert series $(1-t)^{-4}$; in this case, the zero locus, Γ_A, lies in $\mathbb{P}^3 \times \mathbb{P}^3$, where $\mathbb{P}^3 = \mathbb{P}(A_1^*)$. In these cases, Γ_A is the graph of an automorphism $\tau_A : E_A \to E_A$ of a closed subscheme E_A of $\mathbb{P}(A_1^*)$ ([2], [7, Theorem 1.4] and [9, Theorem 1.10]).

One method for constructing a regular algebra makes use of the notion of Ore extension. Before using this method in Section 2, we first prove a technical result about Ore extensions.

Fix a domain R, $\sigma \in \mathrm{Aut}(R)$ and a left σ-derivation, δ, of R. Let S denote the Ore extension $S = R[w; \sigma, \delta]$. Any nonzero element $f \in S$ may be written uniquely as $f = \sum_{i=0}^n a_i w^i$, where each $a_i \in R$ and $a_n \neq 0$. We use $L(m)$ to represent any element of $\sum_{i=0}^m Rw^i$, and we use f^σ to denote $\sigma(f)$, and similarly with f^δ, and $\delta(\sigma(f)) = f^{\sigma\delta}$, etc. In the next result, $Z(R)$ and $Z(S)$ denote the centers of R and S respectively. For $z \in Z(R)$, $C_S(z)$ denotes the centralizer of z in S.

Lemma 1.1. *Let R denote a domain of characteristic zero, and let σ, δ and S be as above. If there exists $z \in Z(R)$ such that $z^\sigma = z$ and $z^{\delta\sigma} = z^\delta \neq 0$, then $Z(S) \subseteq C_S(z) \subseteq R$, and S is an infinite $Z(S)$-module.*

Proof. Since S is an infinite R-module, and since $Z(S) \subseteq C_S(z)$, it suffices to prove that $C_S(z) \subseteq R$. Let $f \in C_S(z)$, $f \neq 0$, and write f uniquely as $f = \sum_{i=0}^n a_i w^i$, with each $a_i \in R$ and $a_n \neq 0$.

A simple computation proves that for any $r \in R$ and $m > 0$,

$$w^m r = r^{\sigma^m} w^m + (r^{\sigma^{m-1}\delta} + r^{\sigma^{m-2}\delta\sigma} + \cdots + r^{\sigma\delta\sigma^{m-2}} + r^{\delta\sigma^{m-1}}) w^{m-1} + L(m-2) \qquad (*)$$

in S. By $(*)$ and the hypotheses on z, for all $m > 0$, we have

$$w^m z = z w^m + m z^\delta w^{m-1} + L(m-2).$$

It follows that

$$\begin{aligned} 0 &= fz - zf \\ &= (a_0 + a_1 w + \cdots + a_n w^n) z - z(a_0 + a_1 w + \cdots + a_n w^n) \\ &= (L(n-2) + a_{n-1} z w^{n-1} + a_n z w^n + a_n n z^\delta w^{n-1}) - (L(n-2) + a_{n-1} z w^{n-1} + a_n z w^n) \\ &= L(n-2) + a_n n z^\delta w^{n-1}. \end{aligned}$$

Thus, $a_n n z^\delta = 0$. However, R is a domain, $z^\delta \neq 0$ and $a_n \neq 0$, so $n = 0$. Since $\mathrm{char}(R) = 0$, this implies $f = a_0 \in R$. ■

2. The Main Theorem and Examples

In this section, we prove our main theorem by giving examples of quadratic, regular algebras of global dimension four that are infinite modules over their centers yet have finite point schemes. We show that the associated automorphisms have finite order, and that each algebra has a one-parameter family of line modules. We assume that the base field, k, has characteristic zero.

Proposition 2.1. *If A is the k-algebra generated by $\{x, y, z, w\}$ subject to the defining relations*

$$\begin{aligned} yx &= -xy, & wx &= -xw + y^2 + ayz + bz^2, \\ zx &= xz, & wy &= -yw + x^2 + cxz + dz^2, \\ zy &= yz, & wz &= zw + xy, \end{aligned}$$

where $a, b, c, d \in k$, then

(a) *A is a noetherian, Artin-Schelter regular domain of global dimension four;*

(b) *A is an infinite module over its center;*

(c) *the point scheme, Γ_A, of A is finite if and only if $b \neq 0$ or $d \neq 0$; if $b \neq 0$ or $d \neq 0$, then Γ_A has at most five closed points, and, if k is algebraically closed, then Γ_A has exactly 2, 3, 4 or 5 closed points (and all four possibilities occur);*

(d) *the automorphism τ_A of E_A has order two;*

(e) *if at least one of a, b, c or d is nonzero, then A has a one-parameter family of line modules.*

Proof. (a) Let R be the subalgebra of A generated by $\{x, y, z\}$. The assignments $x^\sigma = -x$, $y^\sigma = -y$ and $z^\sigma = z$ extend to an automorphism of R. We will show that the assignments

$$x^\delta = y^2 + ayz + bz^2, \qquad y^\delta = x^2 + cxz + dz^2, \qquad z^\delta = xy,$$

extend to a left σ-derivation of R. It suffices to check that δ is well defined on R as follows.

$$\begin{aligned}
(xy + yx)^\delta &= x^\sigma y^\delta + x^\delta y + y^\sigma x^\delta + y^\delta x \\
&= -x(x^2 + cxz + dz^2) + (y^2 + ayz + bz^2)y + \\
&\qquad -y(y^2 + ayz + bz^2) + (x^2 + cxz + dz^2)x \\
&= 0 \quad \text{in } R. \\
(zx - xz)^\delta &= z^\sigma x^\delta + z^\delta x - x^\sigma z^\delta - x^\delta z \\
&= z(y^2 + ayz + bz^2) + (xy)x + x(xy) - (y^2 + ayz + bz^2)z \\
&= 0 \quad \text{in } R. \\
(zy - yz)^\delta &= z^\sigma y^\delta + z^\delta y - y^\sigma z^\delta - y^\delta z \\
&= z(x^2 + cxz + dz^2) + (xy)y + y(xy) - (x^2 + cxz + dz^2)z \\
&= 0 \quad \text{in } R.
\end{aligned}$$

Thus, $A \cong R[w; \sigma, \delta]$. Since R is a noetherian, Artin-Schelter regular (and Auslander-regular) domain of global dimension three that satisfies the Cohen-Macaulay property, it follows from the lemma in [5] that A is an Artin-Schelter regular (and Auslander-regular) noetherian domain that satisfies the Cohen-Macaulay property.

(b) By (a) and Lemma 1.1, A is an infinite module over its center.

(c) If $b = 0 = d$, then $(p, p) \in \Gamma_A$ for all $p \in \mathcal{V}(x, y) \subset \mathbb{P}^3$, so the point scheme is infinite. On the other hand, as discussed in Section 1, Γ_A is isomorphic to the scheme $E_A \subseteq \mathbb{P}^3$. Using homogeneous coordinates (x, y, z, w) on $\mathbb{P}^3$, the scheme E_A is the zero locus in $\mathbb{P}^3$ of the 4×4 minors of the 6×4 matrix:

$$\begin{bmatrix}
y & x & 0 & 0 \\
z & 0 & -x & 0 \\
0 & z & -y & 0 \\
w & -y & -ay - bz & x \\
-x & w & -cx - dz & y \\
0 & -x & w & -z
\end{bmatrix}.$$

If $b \neq 0$ or $d \neq 0$, then E_A (and the point scheme) is finite, and the closed points of E_A are $(0, 0, 0, 1)$, $(0,\ 2,\ 2\alpha,\ d\alpha^2)$ and $(2,\ 0,\ 2\beta,\ b\beta^2)$ for all α and β that are solutions of the equations

$$1 + a\alpha + b\alpha^2 = 0 \qquad \text{and} \qquad 1 + c\beta + d\beta^2 = 0$$

respectively. Thus, the point scheme of A has at most five closed points. If k is algebraically closed, then by choosing a, b, c, d appropriately, we can force the point scheme of A to have exactly 2, 3, 4 or 5 closed points.

(d) The defining relations of A may be written as

$$\begin{array}{rclrcl} yx + xy &=& 0, & wx + xw &=& y^2 + (a/2)yz + (a/2)zy + bz^2, \\ zx - xz &=& 0, & wy + yw &=& x^2 + (c/2)xz + (c/2)zx + dz^2, \\ zy - yz &=& 0, & wz - zw &=& (1/2)xy - (1/2)yx, \end{array}$$

which proves that $A \cong A^{\mathrm{op}}$. Thus, if we consider the morphism, $\eta : \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^3 \times \mathbb{P}^3$, that switches the factors of $\mathbb{P}^3 \times \mathbb{P}^3$, we find $\eta(\Gamma_A) = \Gamma_A$ (since Γ_A is the zero locus of the defining relations of A). This implies that Γ_A is not only the graph of τ_A, but is also the graph of $(\tau_A)^{-1}$. Hence, $\tau_A = (\tau_A)^{-1}$, and so $(\tau_A)^2 = 1$. If $b \neq 0$ or $d \neq 0$, then Γ_A is finite, so we may apply [8, Theorem 4.1], which yields $\tau_A \neq 1$, since A is non-commutative. On the other hand, if $b = 0 = d$, then $(p, p) \in \Gamma_A$ for all $p \in \mathcal{V}(x, y) \in \mathbb{P}^3$. Localizing to the affine open subset of Γ_A where w is nonzero, one easily finds that $\tau_A(z) \neq z$ in the appropriate coordinate ring, and hence $\tau_A \neq 1$. Thus, in both cases, τ_A has order two.

(e) By [8], to determine the size of the family of line modules over A, it suffices to compute the dimension of the scheme of elements of rank at most two in the projectivization of the span of the defining relations of A. To do this, we identify $A_1 \otimes A_1$ with $M_4(k)$ by taking $\{x, y, z, w\}$ to be its own dual basis. Using the "symmetric" form of the defining relations given in the proof of (d) and using coefficients $t_1, \ldots, t_6$, we represent an arbitrary element in the span of the defining relations of A by the 4×4 matrix

$$\begin{bmatrix} -t_5 & t_1 - (1/2)t_6 & -t_2 - (1/2)ct_5 & t_4 \\ t_1 + (1/2)t_6 & -t_4 & -t_3 - (1/2)at_4 & t_5 \\ t_2 - (1/2)ct_5 & t_3 - (1/2)at_4 & -bt_4 - dt_5 & -t_6 \\ t_4 & t_5 & t_6 & 0 \end{bmatrix}.$$

Identifying $(t_1, \ldots, t_6)$ with homogeneous coordinates on $\mathbb{P}^5$, the line scheme is the zero locus of the 3×3 minors of this matrix. A computation shows that if $a = b = c = d = 0$, then the family of line modules has a two-dimensional component (the other components having dimension one); otherwise, all the components have dimension one. ■

The next example has many properties similar to those of the algebra in Proposition 2.1. We include this example since the subject of regular algebras is lacking candidates for *generic* regular algebras of global dimension four on which conjectures may be tested.

Proposition 2.2. *If A is the k-algebra generated by $\{x, y, z, w\}$ subject to the defining relations*

$$\begin{aligned} yx &= -xy + 2z^2, & wx &= -xw + y^2 + ayz + bz^2 + \lambda xz, \\ zx &= xz, & wy &= -yw + x^2 + cxz + dz^2 + (\lambda - 2)yz, \\ zy &= yz, & wz &= zw + xy - z^2, \end{aligned}$$

where $a, b, c, d, \lambda \in k$, then

(a) *A is a noetherian, Artin-Schelter regular domain of global dimension four;*

(b) *A is an infinite module over its center;*

(c) *the point scheme, Γ_A, of A has at most seven closed points; in particular, if k is algebraically closed and if $a = b = c = d = 0$, then Γ_A has exactly seven closed points;*

(d) *the automorphism τ_A of E_A has order two;*

(e) *A has a one-parameter family of line modules.*

Proof. The proof mimics that of Proposition 2.1, so is left to the reader. ■

Each of Propositions 2.1 and 2.2 implies our main theorem given in the Introduction.

The finite-dimensional simple modules over the algebras in Propositions 2.1 and 2.2 have been classified by P. Goetz and B. Shelton in [4].

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