Generalizing Classical Clifford Algebras, Graded Clifford Algebras and their Associated Geometry

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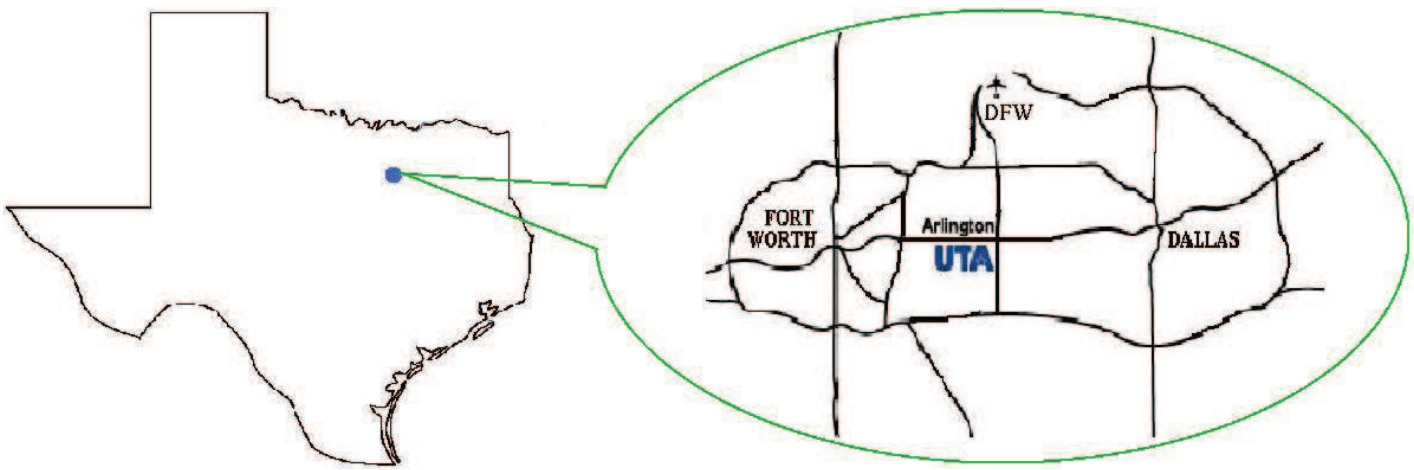

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Graded Clifford Algebras

Throughout this talk, unless otherwise stated,

$\mathbb{k}$ denotes a field where $\text{char}(\mathbb{k}) \neq 2$, and

$M(n, \mathbb{k})$ denotes the ring of $n \times n$ matrices with entries in $\mathbb{k}$.

Whenever any geometry is discussed, assume $\mathbb{k}$ is also algebraically closed.

Definition ([Van den Bergh; Le Bruyn])

Let $M_1, \ldots, M_n \in M(n, \Bbbk)$ denote symmetric matrices. The ($\mathbb{Z}$-)graded Clifford algebra $C = C(M_1, \ldots, M_n)$, associated to $M_1, \ldots, M_n$, is defined to be the associative $\mathbb{Z}$-graded $\Bbbk$-algebra on degree-1 generators $x_1, \ldots, x_n$ and on degree-2 generators $y_1, \ldots, y_n$ with defining relations given by

(a) $x_i x_j + x_j x_i = \sum_{k=1}^{n} (M_k)_{ij}\, y_k$ for all $i, j = 1, \ldots, n$, and

(b) the subalgebra generated by $y_1, \ldots, y_n$ is a polynomial ring contained in the center of C.

Example ($n = 2$)

$$M_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix},$$

$$2x_1^2 = 2y_1, \quad 2x_2^2 = 2y_2,$$
$$x_1x_2 + x_2x_1 = 1y_1 + 0y_2 \; (= x_1^2),$$

so $$C = \frac{\Bbbk\langle x_1, x_2\rangle}{\langle x_1x_2 + x_2x_1 - x_1^2\rangle},$$

where $\Bbbk\langle x_1, x_2\rangle$ is the free algebra generated by x_1 and x_2.

GCAs are noetherian & have some associated geometry as follows.

Example ($n = 2$, previous example, revisit)

$$M_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \leftrightarrow q_1 = 2(t_1^2 + t_1 t_2), \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \leftrightarrow q_2 = 2t_2^2.$$

$\mathcal{Z}(q_1) \cap \mathcal{Z}(q_2) = \emptyset \subset \mathbb{P}^1$. Next theorem $\Rightarrow C = \dfrac{\Bbbk\langle x_1, x_2\rangle}{\langle x_1x_2 + x_2x_1 - x_1^2\rangle}$.

Theorem ([Aubry & Lemaire; Le Bruyn])

The GCA C is quadratic, Auslander regular of global dimension n and satisfies the Cohen-Macaulay property with Hilbert series equal to that of the polynomial ring on n variables if and only if the quadric system in $\mathbb{P}^{n-1}$ associated to $M_1, \ldots, M_n$ is base-point free; in this case, C is a domain.

T. Cassidy and I wanted to extend the above theorem to a larger class of algebras. To do this, we need to generalize the idea of symmetric matrix, quadratic form, quadric system, and.......

μ-symmetric Matrices

Definition ([Cassidy & Vancliff] $\mathbb{k}$ = arbitrary field)

Let $\mu = (\mu_{ij}) \in M(n, \mathbb{k})$ be such that $\mu_{ij}\mu_{ji} = 1$ for all i, j such that $i \neq j$. A matrix $M \in M(n, \mathbb{k})$ is called *μ-symmetric* if $M_{ij} = \mu_{ij} M_{ji}$ for all $i, j = 1, \ldots, n$.

Clearly,

$\mu_{ij} = 1$ for all $i, j \Rightarrow$ μ-symmetric = symmetric

$\mu_{ij} = -1$ for all $i, j \Rightarrow$ μ-symmetric = skew-symmetric ($\text{char}(\mathbb{k}) \neq 2$).

Example

$n = 3$: $\begin{bmatrix} a & b & c \\ \mu_{21} b & d & e \\ \mu_{31} c & \mu_{32} e & f \end{bmatrix}$ is μ-symmetric.

Assumption For the rest of the talk, assume $\mu_{ii} = 1 \;\; \forall\, i$.

Graded Skew Clifford Algebras

Definition ([Cassidy & Vancliff])

With μ as above, let $M_1, \dots, M_n \in M(n, \Bbbk)$ denote μ-symmetric matrices. A *graded skew Clifford algebra*, associated to μ, $M_1, \dots, M_n$, is an associative $\mathbb{Z}$-graded $\Bbbk$-algebra A on degree-1 generators $x_1, \dots, x_n$ and on degree-2 generators $y_1, \dots, y_n$ with defining relations given by:

(a) $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^{n} (M_k)_{ij} y_k$ for all $i, j = 1, \dots, n$, and

(b) the existence of a normalizing sequence $\{y'_1, \dots, y'_n\}$ consisting of homogeneous degree-2 elements of A that span $\Bbbk y_1 + \cdots + \Bbbk y_n$.

(Here, *normalizing* means that, for each i, the image of y'_i in $A/\langle y'_1, \dots, y'_{i-1}\rangle$ is nonzero and the left ideal generated by it equals the right ideal generated by it.)

Clearly, GCAs are GSCAs.

Example ($n = 2$: quantum affine plane)

Let $M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$.

$$2x_1^2 = 2y_1, \quad 2x_2^2 = 2y_2,$$
$$x_1x_2 + \mu_{12}x_2x_1 = 0,$$

so here the GSCA $= \dfrac{\Bbbk\langle x_1, x_2\rangle}{\langle x_1x_2 + \mu_{12}x_2x_1\rangle}$, so is quadratic.

Example (skew polynomial rings)

Similarly, have generators $x_1, \dots, x_n$ with defining relations $x_ix_j + \mu_{ij}x_jx_i = 0$, for all $i \neq j$, are GSCAs. $(M_k = 2E_{kk}\ \forall\ k)$

Example ($n = 2$: Jordan plane)

Let $\mu = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $M_1 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$,

$$2x_1^2 = 2y_1, \quad 2x_2^2 = 2y_2,$$
$$x_1x_2 - x_2x_1 = y_1 = x_1^2,$$

norm seq $= \{x_1^2,\ x_2^2\}$, so the GSCA $= \dfrac{\Bbbk\langle x_1, x_2\rangle}{\langle x_1x_2 - x_2x_1 - x_1^2\rangle}$, so is quadratic.

Remarks

- For all $i, j = 1, \ldots, n$, we have

$$x_j x_i + \mu_{ji} x_i x_j = \sum_{k=1}^{n} (M_k)_{ji}\, y_k = \sum_{k=1}^{n} \mu_{ji} (M_k)_{ij}\, y_k = \mu_{ji}(x_i x_j + \mu_{ij} x_j x_i).$$

  So, in general, there are $n + \binom{n}{2}$ quadratic relations and, possibly, some nonquadratic relations.

- GSCAs are noetherian.

- Given $\mu, M_1, \ldots, M_n$, the definition does not, in general, determine the GSCA uniquely. However, if the algebra is quadratic, then this data determines the algebra.

It remains to generalize notions of quadatic form and quadric to try to relate properties of GSCAs to some geometry.

Given μ & a μ-symmetric matrix $M \in M(n, \Bbbk)$, associate to this data

1. the skew polynomial ring S on generators $z_1, \ldots, z_n$ with defining relations: $z_j z_i = \mu_{ij} z_i z_j$, for all $i \neq j$, (and write S_d for the homogeneous elements of S of degree d), and
2. the element $\begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix} M \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \in S_2$.

Example ($n = 2$: Jordan plane)

$$\mu = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \qquad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

$$z^T M_1 z = 2z_1^2 + z_1 z_2 - z_2 z_1 = 2z_1^2 + 2z_1 z_2 \in S_2, \qquad z^T M_2 z = 2z_2^2 \in S_2.$$

Geometry??

Apply the defining relations of S and elements of S_2 to elements of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ as follows:

Example

$$(z_j z_i - \mu_{ij} z_i z_j)(\ (a_1, \ldots, a_n),\ (b_1, \ldots, b_n)\ ) = a_j b_i - \mu_{ij} a_i b_j \in \{0, 1\}.$$

$$(z_i z_j + z_k^2)(\ (a_1, \ldots, a_n),\ (b_1, \ldots, b_n)\ ) = a_i b_j + a_k b_k \in \{0, 1\}.$$

Here: $\mathbb{P}^{n-1} = \mathbb{P}(S_1^*)$, so the geometry lives in $\mathbb{P}(S_1^*) \times \mathbb{P}(S_1^*)$.

Definition ([Cassidy & Vancliff])

- We call any (nonzero) element of S_2 a quadratic form.
- We define the quadric, $\mathcal{Z}(q)$, determined by any quadratic form q to be the set of points in $\mathbb{P}(S_1^*) \times \mathbb{P}(S_1^*)$ on which q **and** the defining relations of S vanish.

Quadric system and base points???

In the commutative case, the base points of a quadric system Q in $\mathbb{P}^{n-1}$ are parametrized by graded modules $N = \bigoplus_{i=0}^{\infty} N_i$ over $\dfrac{\Bbbk[t_1, \ldots, t_n]}{\langle q : \mathcal{Z}(q) \in Q \rangle}$ that are cyclic (generated by N_0) and satisfy $\dim_{\Bbbk}(N_i) = 1$ for all i.

Hence, Q is base-point free iff there are no such graded modules.

We will use certain modules analogous to these graded modules to define a base point in the noncommutative setting.

Definition ([Cassidy & Vancliff])

- If $q_1, \ldots, q_m \in S_2 \setminus \{0\}$, we call their span a quadric system.
- A quadric system Q is said to be normalizing if it is given by a normalizing sequence of S.
- A right base point of a quadric system Q is a graded right $\frac{S}{\langle Q \rangle}$-module $N = \bigoplus_{i=0}^{\infty} N_i$ that satisfies
  - (a) $N = N_0 \frac{S}{\langle Q \rangle}$, and
  - (b) $\exists\ c \in \mathbb{N} \setminus \{0\}$ such that $\dim_{\Bbbk}(N_i) = c$ for all i, and
  - (c) $\dim_{\Bbbk}(N/N') < \infty$ for all nonzero (graded) submodules N' of N.
- Similarly for left base points.
- We say a quadric system is right base-point free (RBPF) if it has no right base points; similarly, for left base-point free (LBPF).

Remarks

- For right base points of a quadric system that have $c = 1$ in condition (b) of the definition, condition (c) is redundant.
- In the commutative case, since $\Bbbk$ is alg. closed, a right base point corresponds to a base point in the usual sense (localization + nullstellensatz $\Rightarrow c = 1 \Rightarrow N$ cyclic & (c) redundant).
- A normalizing quadric system Q is RBPF iff it is LBPF; in this case we say Q is base-point free (BPF).
- If $c = 1$, we call the module N a point module over $S/\langle Q \rangle$.
- If Q is a normalizing quadric system, then the isomorphism classes of point modules over $S/\langle Q \rangle$ are parametrized by the points of $\bigcap_{q \in Q} \mathcal{Z}(q)$.

We now recall the theorem we wish to generalize from GCAs to GSCAs.

Theorem ([Aubry & Lemaire; Le Bruyn])

The GCA C is quadratic, Auslander regular of global dimension n and satisfies the Cohen-Macaulay property with Hilbert series equal to that of the polynomial ring on n variables if and only if the quadric system in $\mathbb{P}^{n-1}$ associated to $M_1, \ldots, M_n$ is base-point free; in this case, C is a domain.

Theorem ([Cassidy & Vancliff])

Let μ be as above and suppose $M_1, \ldots, M_n \in M(n, \Bbbk)$ are μ-symmetric matrices.

A GSCA $A = A(\mu, M_1, \ldots, M_n)$ is quadratic, Auslander regular of global dimension n and satisfies the Cohen-Macaulay property with Hilbert series equal to that of the polynomial ring on n variables if and only if the quadric system associated to $M_1, \ldots, M_n$ is normalizing and base-point free; in this case, A is a domain.

Example ($n = 2$: quantum affine plane)

$$M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \quad \Rightarrow Q = \Bbbk z_1^2 + \Bbbk z_2^2 \subset S_2, \quad S = \frac{\Bbbk\langle z_1, z_2\rangle}{\langle z_2 z_1 - \mu_{12} z_1 z_2\rangle}$$

and both z_1^2 and z_2^2 are normal in S, so Q is a normalizing quadric system.

Any base points?

Suppose $N = \bigoplus_{i=0}^{\infty} N_i = N_0 \frac{S}{\langle Q\rangle}$ is a base point.

$\dim_{\Bbbk}(N) = \infty \Rightarrow \dim_{\Bbbk}(\frac{S}{\langle Q\rangle}) = \infty$, however,

$$\frac{S}{\langle Q\rangle} = \frac{\Bbbk\langle z_1, z_2\rangle}{\langle z_2 z_1 - \mu_{12} z_1 z_2, \ z_1^2, \ z_2^2\rangle}, \quad \text{which has dimension } 4 \neq \infty.$$

Hence, Q is normalizing and BPF, so the GSCA is quadratic, Auslander regular etc.

Example ($n = 2$: Jordan plane)

$$\mu = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad S = \frac{\Bbbk\langle z_1, z_2\rangle}{\langle z_2 z_1 + z_1 z_2\rangle}.$$

$\Rightarrow Q = \Bbbk(z_1^2 + z_1 z_2) + \Bbbk z_2^2 \subset S_2.$

$\{z_2^2,\ z_1^2 + z_1 z_2\}$ is a normalizing sequence in S since

$(z_1^2 + z_1 z_2) z_1 = (z_1 - 2z_2)(z_1^2 + z_1 z_2)$ in $S/\langle z_2^2\rangle$, and

$z_1(z_1^2 + z_1 z_2) = (z_1^2 + z_1 z_2)(z_1 + 2z_2)$ in $S/\langle z_2^2\rangle$, and

$(z_1^2 + z_1 z_2) z_2 = z_2(z_1^2 + z_1 z_2)$ in $S/\langle z_2^2\rangle$.

Hence, Q is a normalizing quadric system.

$\frac{S}{\langle Q\rangle}$ has dimension 4, so Q has no base points (by using the argument provided in the previous example).

Thus, Q is normalizing and BPF, so the GSCA is quadratic, Auslander regular etc.

Definition

If B is a finitely generated, quadratic algebra, then we can write $B = \frac{T(V)}{\langle W \rangle}$, where $T(V)$ denotes the tensor algebra on a finite-dimensional vector space V, and W is a subspace of $V \otimes_{\Bbbk} V$. The Koszul dual of B is the algebra $B^! = \dfrac{T(V^*)}{\langle W^{\perp} \rangle}$, where $W^{\perp} \subset V^* \otimes_{\Bbbk} V^*$.

For a quadratic GSCA, A, we have that $A^! \cong \dfrac{S}{\langle Q \rangle}$.

Example ($n = 2$: Jordan plane)

$V = \Bbbk x_1 \oplus \Bbbk x_2, \quad W = \Bbbk(x_1 \otimes x_2 - x_2 \otimes x_1 - x_1 \otimes x_1).$

$S_1 = \Bbbk z_1 \oplus \Bbbk z_2, \quad$ where $\{z_1, z_2\} =$ dual basis to $\{x_1, x_2\}$.

$W^{\perp} = \Bbbk(z_2 \otimes z_1 + z_1 \otimes z_2) \oplus \Bbbk(z_2 \otimes z_2) \oplus \Bbbk(z_1 \otimes z_1 + z_1 \otimes z_2).$

Skew Clifford Algebras

Until now, our discussion has centered on GSCAs, which are $\mathbb{Z}$-graded algebras that can be viewed as quantized analogues of GCAs.

Since a GCA maps onto a Clifford algebra, is there a quantized/skew analogue of this situation?

The rest of this talk will address this, but keep in mind that the algebra formally defined on the next slide can, depending on the input data, turn out to be $\{0\}$ . . . . . .

Example

$$\frac{\Bbbk\langle x_1, x_2\rangle}{\langle x_1^2, x_2^2, x_1x_2 + x_2x_1 - 1\rangle}$$

is a Clifford algebra of dim $= 4$

vs.

$$\frac{\Bbbk\langle x_1, x_2\rangle}{\langle x_1^2, x_2^2, x_1x_2 - x_2x_1 - 1\rangle} = \{0\}$$

is <u>not</u> a Clifford algebra.

Quantized analogues exist that use various conditions (e.g., Chen & Kang use bicharacters), but the definition below only uses μ-symmetry.

Definition ([Cassidy & Vancliff])

(a) Let V be a vector space with basis $\mathcal{B} = \{x_1, \ldots, x_n\}$ and let μ be as above. We call a bilinear form $\phi : V \times V \to \Bbbk$ μ-symmetric (relative to $\mathcal{B}$) if $\phi(x_i, x_j) = \mu_{ij}\phi(x_j, x_i)$ for all i, j.

(b) Let V be a vector space with ordered basis $\mathcal{B} = \{x_1, \ldots, x_n\}$, and let ϕ be a μ-symmetric bilinear form (relative to $\mathcal{B}$). The skew Clifford algebra $\mathrm{sCl}(V, \mu, \phi)$ associated with ϕ is the quotient of the tensor algebra on V by the ideal generated by all elements of the form $x_i \otimes x_j + \mu_{ij} x_j \otimes x_i - 2\phi(x_i, x_j) \cdot 1$ for all i, j.

Example

$$V = \Bbbk^2, \qquad \mu = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad (\phi(x_i, x_j)) = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix},$$

$\Rightarrow \mathrm{sCl}(V, \mu, \phi) = \{0\}$.

Example Any Clifford algebra is a skew Clifford algebra.

Example $\phi = 0 \Rightarrow \mathrm{sCl}(V, \mu, 0) = \Lambda_\mu(V) =$ the quantum exterior algebra.

$\dim_{\Bbbk}(\mathrm{sCl}(V, \mu, 0)) = 2^{\dim(V)}$.

Example

$$\dim_{\Bbbk}(V) = 3, \qquad \mu = \begin{bmatrix} 1 & a & 1 \\ 1/a & 1 & a \\ 1 & 1/a & 1 \end{bmatrix}, \qquad (\phi(x_i, x_j)) = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ b & 0 & 0 \end{bmatrix},$$

where $a, b \in \Bbbk$ such that $a \neq 0 \Rightarrow \dim_{\Bbbk}(\mathrm{sCl}(V, \mu, \phi)) = 8$.

Example

$$\dim_{\Bbbk}(V) = 4, \qquad \mu = \begin{bmatrix} 1 & \mu_{12} & \mu_{13} & 1 \\ \mu_{21} & 1 & \mu_{23} & 1 \\ \mu_{31} & \mu_{32} & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad (\phi(x_i, x_j)) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & c \\ a & b & c & 1 \end{bmatrix},$$

where $a, b, c \in \Bbbk$, such that $abc \neq 0$. If $\mu_{23} = \mu_{13} = 1 \neq \mu_{12}$, then $\dim_{\Bbbk}(\mathrm{sCl}(V, \mu, \phi)) = 8$. If $\mu_{23} \neq 1$ or $\mu_{13} \neq 1$, then $\dim_{\Bbbk}(\mathrm{sCl}(V, \mu, \phi)) = 4$.

Let $g : V \to \mathsf{sCl}(V, \mu, \phi)$ denote the composition

$$V \hookrightarrow T(V) \twoheadrightarrow \mathsf{sCl}(V, \mu, \phi).$$

For Clifford algebras, g is always injective, but, as seen above, for arbitrary $\mathsf{sCl}(V, \mu, \phi)$, g need not be injective.
Nevertheless, using g, there is a universal mapping property analogous to that for Clifford algebras.

The quantum exterior algebra $\Lambda_\mu(V) = \mathsf{sCl}(V, \mu, 0)$ is quadratic, so has a Koszul dual, $\Lambda_\mu(V)^!$.
In fact, $\Lambda_\mu(V)^!$ is the algebra S from the 1st half of this talk, where S is generated by $\{z_1, \ldots, z_n\}$ which is the dual basis to $\{x_1, \ldots, x_n\}$.
Hence, we can build a quadratic form in S_2 as described above by using ϕ:

$$\begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix} (\phi(x_i, x_j)) \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = q \in S_2.$$

Theorem ([Cassidy & Vancliff])

The following are equivalent:

(a) *the map* $g : V \to \mathrm{sCl}(V, \mu, \phi)$ *is injective;*

(b) $\dim_{\Bbbk}(\mathrm{sCl}(V, \mu, \phi)) = 2^{\dim(V)}$;

(c) *in* $S = \Lambda_{\mu}(V)^! = \mathrm{sCl}(V, \mu, 0)^!$, *the element* q *determined by* ϕ *is central.*

Theorem ([Cassidy & Vancliff])

If $\mathrm{sCl}(V, \mu, \phi) \neq \{0\}$, *then*

$$\dim_{\Bbbk}(\mathrm{sCl}(V, \mu, \phi)) = 2^{j},$$

where $1 \leq j \leq \dim_{\Bbbk}(V)$.

Relationship between GSCAs and skew Clifford algebras?

Theorem ([Cassidy & Vancliff])

Write $n = \dim_{\Bbbk}(V)$ *and suppose* $\phi \neq 0$.

(a) $\mathrm{sCl}(V, \mu, \phi)$ *is a quotient of a GSCA that is generated by degree-1 elements.*

(b) *If* $\dim_{\Bbbk}(\mathrm{sCl}(V, \mu, \phi)) = 2^n$, *then there exists a quadratic Auslander-regular GSCA associated to* μ *and* $M_1, \ldots, M_n$ *that maps onto* $\mathrm{sCl}(V, \mu, \phi)$ *if and only if* $\mu_{ij}^2 = 1$ *for all* i, j.

Why GSCAs?

In the study of Auslander-regular algebras and in the study of Artin-Schelter regular algebras, it was an open question for many years whether or not there could exist a quadratic regular algebra of global dimension 4 that has finitely many point modules and a 1-parameter family of line modules.

Various results imply that such algebras are supposed to be plentiful, but none were known for many years.

In 2000, one (and only one) such algebra was found by Shelton and Tingey. Their method was trial and error at a computer, and they were not able to duplicate their methods to find more than one such algebra.

In contrast, it was known that there exist GCAs that satisfy almost all the needed properties, but their line modules are parametrized by 2 (or more) parameters.

Cassidy and I believed that it should be possible to "skew" the symmetry of a GCA a small amount, in such a way that the desirable properties are retained, while reducing the number of line modules to a 1-parameter family.

We were successful in this goal: we found families of quadratic regular GSCAs of global dimension 4 that have finitely many point modules and a 1-parameter family of line modules.

References

- M. Aubry & J.-M. Lemaire, Zero Divisors in Enveloping Algebras of Graded Lie Algebras, *J. of Pure & App. Alg.* **38** (1985), 159-166.
- T. Cassidy & M. Vancliff, Generalizations of Graded Clifford Algebras and of Complete Intersections, *J. Lond. Math. Soc.* **81** (2010), 91-112.
(Corrigendum: **90** No. 2 (2014), 631-636.)
- T. Cassidy & M. Vancliff, Skew Clifford Algebras, *J. of Pure & App. Alg.* **223** No. 12 (2019), 5091-5105.
- Z. Chen & Y. Kang, Generalized Clifford Theory for Graded Spaces, *J. of Pure & App. Alg.* **220** No. 2 (2016), 647-665.
- L. Le Bruyn, Central Singularities of Quantum Spaces, *J. Alg.* **177** (1995), 142-153.
- B. Shelton & C. Tingey, On Koszul Algebras and a New Construction of Artin–Schelter Regular Algebras, *J. Alg.* **241** (2001), 789-798.

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