THE QUANTUM SPACES OF CERTAIN GRADED ALGEBRAS RELATED TO $\mathfrak{sl}(2, \Bbbk)$

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Abstract. Inspired by the work of Le Bruyn and Smith in [12] and the work of Shelton and Vancliff in [17], we analyze certain graded algebras related to the Lie algebra $\mathfrak{sl}(2, \Bbbk)$ using geometric techniques in the spirit of Artin, Tate and Van den Bergh. In particular, we discuss the point schemes and line schemes of certain quadratic quantum $\mathbb{P}^3$s associated to the Lie superalgebra $\mathfrak{sl}(1|1)$, to a quantized enveloping algebra, $\mathcal{U}_q(\mathfrak{sl}(2, \Bbbk))$, of $\mathfrak{sl}(2, \Bbbk)$, and to a color Lie algebra $\mathfrak{sl}_k(2, \Bbbk)$, respectively. The geometry we consider identifies certain normal elements in the universal enveloping algebra of $\mathfrak{sl}(1|1)$ and in $\mathcal{U}_q(\mathfrak{sl}(2, \Bbbk))$.

Introduction

A regular algebra of global dimension four is often viewed as a noncommutative analogue of a polynomial ring on four variables. In the language of [4], such an algebra is often called a quantum $\mathbb{P}^3$.

In [12], the point modules and line modules, called the quantum space, of the homogenization $\mathcal{H}(\mathfrak{sl}(2, \Bbbk))$ of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}(2, \Bbbk)$ were determined.

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Therein, it was shown that the scheme of points in $\mathbb{P}^3$ that correspond to the point modules of $\mathcal{H}(\mathfrak{sl}(2,\Bbbk))$ is the union of a point, a plane, and a conic embedded in the plane, whereas the lines in $\mathbb{P}^3$ that correspond to the line modules of $\mathcal{H}(\mathfrak{sl}(2,\Bbbk))$ are the lines lying on a certain pencil of quadrics. The quantum space of $\mathcal{H}(\mathfrak{sl}(2,\Bbbk))$ identifies certain features of $\mathfrak{sl}(2,\Bbbk)$, including the existence of a Casimir element in the universal enveloping algebra.

The definition of line scheme is given in [16]. In [17], a method was given for computing the line scheme of any quadratic algebra on four generators that is a domain and has a Hilbert series the same as that of the polynomial ring on four variables. Therein, it was shown that the line scheme of $\mathcal{H}(\mathfrak{sl}(2,\Bbbk))$ is neither irreducible nor reduced and has two two-dimensional components. In contrast, the computation of the line modules in [12] was done without the use of the line scheme.

Following the work in [12, 17], we analyze a quantum space associated to the Lie superalgebra $\mathfrak{sl}(1|1)$ (cf. [6]), to a quantized enveloping algebra, $\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk))$, of $\mathfrak{sl}(2,\Bbbk)$ (cf. [18]), and to the color Lie algebra $\mathfrak{sl}_k(2,\Bbbk)$ defined in [5]. As in [12], the quantum space we associate to each of the first two algebras identifies some distinguished elements in the associated graded algebra.

The article is outlined as follows. In Section 1, we discuss some background material relevant to later sections. In Section 2, we analyze $\mathfrak{sl}(1|1)$ by first computing the point scheme and line variety of an associated quadratic quantum $\mathbb{P}^3$, denoted $\mathcal{H}(\mathfrak{sl}(1|1))$. We then identify the lines in $\mathbb{P}^3$ that correspond to the line modules of $\mathcal{H}(\mathfrak{sl}(1|1))$ and then identify $\mathcal{H}(\mathfrak{sl}(1|1))$ as a twist by an automorphism of a member of the family of coordinate rings of quantum 2×2 matrices ([8]); this aids in recognizing a certain homogeneous degree-2 element of $\mathcal{H}(\mathfrak{sl}(1|1))$ as a distinguished normal (supercommuting) element. In Section 3, we analyze $\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk))$ in a similar manner using an associated quadratic quantum $\mathbb{P}^3$, denoted $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$. The geometry identifies an element in $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ related to the quantum Casimir element in $\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk))$. In Section 4, we examine $\mathfrak{sl}_k(2,\Bbbk)$ via an associated quadratic quantum $\mathbb{P}^3$, denoted $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$. The geometry associated to $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$ suggests there is no distinguished central, or normal, element in $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$, unlike the analysis of $\mathcal{H}(\mathfrak{sl}(2,\Bbbk))$, $\mathcal{H}(\mathfrak{sl}(1|1))$ and $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$.

1. Preliminaries

Throughout this article, $\Bbbk$ denotes an algebraically closed field of characteristic zero, $\Bbbk^\times = \Bbbk \setminus \{0\}$, $\mathbb{N}$ denotes the set of nonnegative integers, $\mathcal{V}(f_1,\ldots,f_m)$ denotes the zero locus of the polynomials $f_1,\ldots,f_m$ and $e_1,\ldots,e_4$ denote the points $e_1=(1,0,0,0)$, $e_2=(0,1,0,0)$, $e_3=(0,0,1,0)$ and $e_4=(0,0,0,1)$ in $\mathbb{P}^3$.

Given a quadratic regular algebra, A, on degree-one generators x_1,x_2,x_3,x_4, we may follow [1] in order to compute the point scheme of A. We write the defining relations of A in the form

$Mx = 0$, where M is 6×4 matrix and x is a column vector given by $x^T = (x_1,\, x_2,\, x_3,\, x_4)$. The point scheme of A can be identified with the zero locus in $\mathbb{P}(A_1^*) \cong \mathbb{P}^3$ of the 4×4 minors of M.

Our computation of the line scheme as a subscheme of $\mathbb{P}^5$ will make use of the Plücker coordinates, M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34} on $\mathbb{P}^5$. We first recall how these coordinates relate to lines in $\mathbb{P}^3$; details may be found in [7, §8.6]. Any line ℓ in $\mathbb{P}^3$ is uniquely determined by any two distinct points $(a_1,\, a_2,\, a_3,\, a_4)$ and $(b_1,\, b_2,\, b_3,\, b_4)$ that lie on ℓ; so ℓ may be represented by a 2×4 matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

that has rank two; in particular, the points on ℓ are represented in homogeneous coordinates by linear combinations of the rows of this matrix. The Plücker coordinate M_{ij} is evaluated on this matrix as the minor $a_i b_j - a_j b_i$, for all $i < j$, and the polynomial $P = M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}$ vanishes on this matrix. Moreover, $\mathcal{V}(P)$ is the subscheme of $\mathbb{P}^5$ that parametrizes all the lines in $\mathbb{P}^3$.

2. The Lie Superalgebra $\mathfrak{sl}(1|1)$

2.1. The $\Bbbk$-Algebra $\mathcal{H}(\mathfrak{sl}(1|1))$.

Let $\mathfrak{sl}(1|1) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \Bbbk h$ and $\mathfrak{g}_1 = \Bbbk e \oplus \Bbbk f$, denote the Lie superalgebra (cf. [6]) with bracket defined by

$$[e,\, f] = h, \qquad [h,\, e] = [h,\, f] = [e,\, e] = [f,\, f] = [h,\, h] = 0.$$

The universal enveloping algebra (cf. [6]) of $\mathfrak{sl}(1|1)$ is

$$\mathcal{U}(\mathfrak{sl}(1|1)) = \frac{\Bbbk\langle e,\, f,\, h\rangle}{\langle\ ef + fe - h,\ he - eh,\ hf - fh,\ e^2,\ f^2\ \rangle}.$$

In order to obtain a quadratic quantum $\mathbb{P}^3$ that maps onto $\mathcal{U}(\mathfrak{sl}(1|1))$, we define the following algebra. In Theorem 2.2, we establish that the algebra is indeed a quadratic quantum $\mathbb{P}^3$.

Definition 2.1. The quadratic $\Bbbk$-algebra, $\mathcal{H}(\mathfrak{sl}(1|1))$, on degree-one generators e, f, h, t is

$$\mathcal{H}(\mathfrak{sl}(1|1)) = \frac{\Bbbk\langle e,\, f,\, h,\, t\rangle}{\langle\ ef + fe - ht,\ he - eh,\ hf - fh,\ et - te,\ ft - tf,\ ht - th\ \rangle}.$$

Note that $\mathcal{U}(\mathfrak{sl}(1|1)) \cong \mathcal{H}(\mathfrak{sl}(1|1))/\langle t - 1,\, e^2,\, f^2\rangle$.

Theorem 2.2. *The $\Bbbk$-algebra $\mathcal{H}(\mathfrak{sl}(1|1))$ is a quadratic quantum $\mathbb{P}^3$.*

Proof. Let $A = \Bbbk[e,\, h,\, t]$. The algebra $\mathcal{H}(\mathfrak{sl}(1|1))$ is isomorphic to the Ore extension (cf. [9]) $A[f;\, \phi,\, \delta]$, where ϕ and δ are an automorphism and a left ϕ-derivation, respectively, of A defined by

$$\phi(e) = -e, \qquad \phi(h) = h, \qquad \phi(t) = t, \qquad \delta(e) = ht, \qquad \delta(h) = 0, \qquad \delta(t) = 0.$$

Hence, by [15], $\mathcal{H}(\mathfrak{sl}(1|1))$ is Auslander-regular and satisfies the Cohen-Macaulay property. By the definition of the Cohen-Macaulay property ([14, Definition 5.8]), $\mathcal{H}(\mathfrak{sl}(1|1))$ has polynomial growth and is, hence, AS-regular by [14]. By [14, Theorem 4.8], $\mathcal{H}(\mathfrak{sl}(1|1))$ is a domain and, thus, t is a central regular element of $\mathcal{H}(\mathfrak{sl}(1|1))$. It follows that $\mathcal{H}(\mathfrak{sl}(1|1))$ is a central regular extension of $\Bbbk[e, f, h]$, in the language of [13], and so is AS-regular of global dimension four (cf. [13, Theorem 2.6, Corollary 2.7] and the paragraph after [13, Definition 3.1.1]). Hence, $\mathcal{H}(\mathfrak{sl}(1|1))$ is a quadratic quantum $\mathbb{P}^3$. ■

Note that there exists an automorphism φ on $\mathcal{H}(\mathfrak{sl}(1|1))$ defined by $\varphi : e \leftrightarrow f$.

2.2. **The Quantum Space of $\mathcal{H}(\mathfrak{sl}(1|1))$.**

Denote $x_1 = e$, $x_2 = f$, $x_3 = h$ and $x_4 = t$.

Theorem 2.3. *The point scheme of $\mathcal{H}(\mathfrak{sl}(1|1))$ is isomorphic to the subscheme*

$$\mathfrak{p} = \mathcal{V}(x_3x_4 - 2x_1x_2) \cup \mathcal{V}(x_3,\, x_4)$$

of $\mathbb{P}^3$; that is, $\mathfrak{p}$ is the union of a nonsingular quadric and a line in $\mathbb{P}^3$ that meets the quadric at two distinct points.

Proof. The polynomials that define $\mathfrak{p}$ are listed in [3, §5.2.1]. The zero locus of these polynomials is $\mathcal{V}(x_3x_4 - 2x_1x_2) \cup \mathcal{V}(x_3,\, x_4)$.

The Jacobian matrix, J, of $\mathfrak{p}$ is given in [3, §5.2.2]. If p is a multiple point contained in a d-dimensional, irreducible, component of $\mathfrak{p}$, then the $(3-d)\times(3-d)$ minors of $J|_p$ vanish, and conversely ([10]). The only such points in $\mathfrak{p}$ are $e_1, e_2 \in \mathcal{V}(x_3x_4 - 2x_1x_2) \cap \mathcal{V}(x_3,\, x_4)$.

The coordinate ring of $\mathfrak{p}$ is $\Bbbk[x_1,\, x_2,\, x_3,\, x_4]/I$, where I is the ideal generated by the polynomials in [3, §5.2.1]. The line $L = \mathcal{V}(x_2 - x_3,\, x_4)$ is a linear scheme of complementary dimension to $\mathcal{V}(x_3x_4 - 2x_1x_2)$ in $\mathbb{P}^3$ and the points of intersection of L and $\mathfrak{p}$ are e_1 and $(0, 1, 1, 0)$. Localizing the coordinate ring of $\mathfrak{p} \cap L$ around e_1 yields a ring isomorphic to a polynomial ring on one variable, x, with exactly one relation $x^2 = 0$, and so e_1 has multiplicity two. This implies that e_1 is a multiple point of $\mathfrak{p}$ only as a consequence of being an intersection point of the two irreducible components of $\mathfrak{p}$; applying the automorphism φ implies that the same applies to e_2. ■

Corollary 2.4. *Let $A = \mathcal{H}(\mathfrak{sl}(1|1))$.*

(i) *The points in $\mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$ on which the defining relations of $\mathcal{H}(\mathfrak{sl}(1|1))$ vanish are of the form $(p,\, p)$ if $p \in \mathcal{V}(x_3x_4 - 2x_1x_2)$, and are of the form $((\alpha_1,\, \alpha_2,\, 0,\, 0),\ (\alpha_1,\, -\alpha_2,\, 0,\, 0))$ if $(\alpha_1,\, \alpha_2,\, 0,\, 0) \in \mathcal{V}(x_3,\, x_4)$.*

(ii) *There exists an automorphism* $\sigma : \mathfrak{p} \to \mathfrak{p}$ *which, on closed points, is defined by*

$$\sigma(p) = \sigma(p_1,\, p_2,\, p_3,\, p_4) = \begin{cases} p, & p \in \mathcal{V}(x_3x_4 - 2x_1x_2) \\ (p_1,\, -p_2,\, 0,\, 0), & p \in \mathcal{V}(x_3,\, x_4). \end{cases}$$

Proof. Part (i) is shown by computation. The existence of the map in (ii) follows from (i) and [13]. ■

Theorem 2.5. *The line scheme of* $\mathcal{H}(\mathfrak{sl}(1|1))$ *has dimension three and its reduced variety,* $\mathfrak{L}$*, is the union of the irreducible components*:

(I) $\mathfrak{L}_1 = \mathcal{V}(M_{34},\ M_{13}M_{24} - M_{14}M_{23})$,
(II) $\mathfrak{L}_2 = \mathcal{V}(M_{14},\ M_{23},\ 2M_{12} + M_{34},\ M_{34}^2 + 2M_{13}M_{24})$,
(III) $\mathfrak{L}_3 = \mathcal{V}(M_{13},\ M_{24},\ 2M_{12} - M_{34},\ M_{34}^2 + 2M_{14}M_{23})$.

Proof. For details on the construction of the line scheme from the defining relations of a quadratic quantum $\mathbb{P}^3$, the reader is referred to [4, 17].

The polynomials that define the line scheme are given in [3, §5.2.3]. The reduced variety given by those polynomials is $\mathfrak{L}$. For details of the computation, the reader is referred to [3]. ■

Corollary 2.6. *The lines in* $\mathbb{P}^3$ *that are parametrized by the closed points of the line scheme of* $\mathcal{H}(\mathfrak{sl}(1|1))$ *are*

(i) *the lines in* $\mathbb{P}^3$ *that intersect the line* $\mathcal{V}(x_3,\, x_4)$*, and*
(ii) *the lines in* $\mathbb{P}^3$ *that lie on the quadric* $\mathcal{V}(2x_1x_2 - x_3x_4)$*.*

Proof. Let $(a_1,\, a_2,\, a_3,\, a_4),\ (b_1,\, b_2,\, b_3,\, b_4) \in \mathbb{P}^3$ be distinct points and let

$$\ell = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

represent the line between them.

(i) If ℓ is given by a point of $\mathfrak{L}_1$, then it follows that ℓ is represented by a rank-two matrix of the form

$$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_5 & c_6 & 0 & 0 \end{bmatrix},$$

where $c_1, \ldots, c_6 \in \Bbbk$. Hence, ℓ intersects $\mathcal{V}(x_3,\, x_4)$ if and only if ℓ is given by $\mathfrak{L}_1$.

(ii) If ℓ is given by $\mathfrak{L}_2$, then if follows that ℓ may be represented by

$$\begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_2 & a_3 & 0 \end{bmatrix},$$

where $(a_1,\, a_4),\ (a_2,\, a_3) \in \mathbb{P}^1$ and $2a_1a_2 - a_3a_4 = 0$. Both $(a_1,\, 0,\, 0,\, a_4)$ and $(0,\, a_2,\, a_3,\, 0)$ lie on the quadric $\mathcal{V}(2x_1x_2 - x_3x_4)$; in fact, for any point $p = (a_1,\, \delta a_2,\, \delta a_3,\, a_4) \in \ell$, where $\delta \in \mathbb{P}^1$, p belongs to $\mathcal{V}(2x_1x_2 - x_3x_4)$. Thus, ℓ belongs to one of the rulings of the quadric $\mathcal{V}(2x_1x_2 - x_3x_4)$. In particular, $\ell \in \{\mathcal{V}(\mu x_1 - x_4,\, 2x_2 - \mu x_3) : \mu \in \mathbb{P}^1\}$.

The component $\mathfrak{L}_3$ may be obtained from $\mathfrak{L}_2$ by applying φ; thus, ℓ is given by $\mathfrak{L}_3$ if and only if ℓ belongs to the ruling $\{\mathcal{V}(\mu x_1 - x_3,\, 2x_2 - \mu x_4) : \mu \in \mathbb{P}^1\}$ of $\mathcal{V}(2x_1x_2 - x_3x_4)$.

Since every line on $\mathcal{V}(2x_1x_2 - x_3x_4)$ belongs to exactly one of the rulings above, it follows that the lines on $\mathcal{V}(2x_1x_2 - x_3x_4)$ are in one-to-one correspondence with the points in $\mathfrak{L}_2 \cup \mathfrak{L}_3$. ■

2.3. Twisting $\mathcal{O}_q(\mathbb{M}_2)$.

Given the results of Section 2.2, the quantum space of $\mathcal{H}(\mathfrak{sl}(1|1))$ is isomorphic to the quantum space of the coordinate ring, $\mathcal{O}_q(\mathbb{M}_2)$, of quantum 2×2 matrices (see Definition 2.7). This fact suggests that $\mathcal{H}(\mathfrak{sl}(1|1))$ could be isomorphic to a twist by an automorphism of $\mathcal{O}_q(\mathbb{M}_2)$, for some $q \in \Bbbk^\times$.

Definition 2.7. [8] The coordinate ring of quantum 2×2 matrices is the $\Bbbk$-algebra

$$\mathcal{O}_q(\mathbb{M}_2) = \frac{\Bbbk\langle a,\, b,\, c,\, d\rangle}{\langle ab - qba,\, cd - qdc,\, ac - qca,\, bd - qdb,\, bc - cb,\, ad - da - (q - q^{-1})bc\rangle},$$

where $q \in \Bbbk^\times$ and $q^2 \neq 1$.

Lemma 2.8. [19]

(i) *The point scheme of* $\mathcal{O}_q(\mathbb{M}_2)$ *is* $\mathcal{V}(ad - bc) \cup \mathcal{V}(b,\, c)$.

(ii) *The closed points of the line scheme of* $\mathcal{O}_q(\mathbb{M}_2)$ *parametrize all the lines in* $\mathbb{P}^3$ *that belong to* $\mathcal{V}(ad - bc)$ *and all lines that intersect* $\mathcal{V}(b,\, c)$.

Theorem 2.9. *The algebra* $\mathcal{H}(\mathfrak{sl}(1|1))$ *is isomorphic to a twist by an automorphism (in the sense of* [2, §8]*) of the algebra* $\mathcal{O}_i(\mathbb{M}_2)$*, where* $i^2 = -1$.

Proof. Let $A = \mathcal{O}_i(\mathbb{M}_2)$ and $\tau : A \to A$ be the automorphism defined by

$$\tau(a) = ia, \qquad \tau(b) = b, \qquad \tau(c) = c, \qquad \tau(d) = -id.$$

Let A^τ be the algebra obtained by twisting A by τ, with multiplication, $\star$, defined by $\overline{x} \star \overline{y} = x\tau(y)$, for all $\overline{x},\ \overline{y} \in A^\tau_1$, where $\overline{x},\ \overline{y}$ are the elements of A^τ_1 corresponding to x and y in A_1. It follows that

$$A^\tau \cong \frac{\Bbbk\langle a, b, c, d\rangle}{\langle\ ab - ba,\ ac - ca,\ bc - cb,\ bd - db,\ cd - dc,\ ad + da - 2bc\ \rangle},$$

which is isomorphic to $\mathcal{H}(\mathfrak{sl}(1|1))$ by mapping $a \mapsto e,\ b \mapsto h/\sqrt{2},\ c \mapsto t/\sqrt{2},\ d \mapsto f$. ■

This result leads to the identification of a distinguished supercommuting element of $\mathcal{H}(\mathfrak{sl}(1|1))$ (and, therefore, of $\mathcal{U}(\mathfrak{sl}(1|1))$) as follows. The element $ad - ibc \in \mathcal{O}_i(\mathbb{M}_2)$, called the quantum determinant, is central in $\mathcal{O}_i(\mathbb{M}_2)$; its image in $\mathcal{H}(\mathfrak{sl}(1|1))$ belongs to $\Bbbk^\times(2ef - ht)$, which corresponds to the quadric $\mathcal{V}(2ef - ht)$ in the point scheme of $\mathcal{H}(\mathfrak{sl}(1|1))$, since $2ef - ht$ vanishes on the graph of $\sigma|_{\mathcal{V}(2ef-ht)}$. Regrading $\mathcal{H}(\mathfrak{sl}(1|1))$ by taking e and f to be odd and h and t to be even, $2ef - ht$ supercommutes in the algebra. The image of this element in

$\mathcal{U}(\mathfrak{sl}(1|1))$ is $2ef - h = ef - fe$ which supercommutes in $\mathcal{U}(\mathfrak{sl}(1|1))$. More precisely, $\mathcal{U}(\mathfrak{sl}(1|1))$ has a PBW basis given by $\{e^i f^j e^k : i,\ j,\ k \in \mathbb{N}\}$ and can be realized as a superalgebra by taking

$$\mathcal{U}(\mathfrak{sl}(1|1))_0 = \bigoplus_{i+j+k \text{ is even}} \Bbbk\left(e^i f^j h^k\right), \qquad \mathcal{U}(\mathfrak{sl}(1|1))_1 = \bigoplus_{i+j+k \text{ is odd}} \Bbbk\left(e^i f^j h^k\right).$$

If $x \in \mathcal{U}(\mathfrak{sl}(1|1))_0$, then $x(2ef - h) = (2ef - h)x$; if $x \in \mathcal{U}(\mathfrak{sl}(1|1))_1$, then $x(2ef - h) = -(2ef - h)x$.

Hence, one may view the element $2ef - h \in \mathcal{U}(\mathfrak{sl}(1|1))$ as playing the role of a Casimir element of a Lie superalgebra, which is not a well-defined concept for $\mathcal{U}(\mathfrak{sl}(1|1))$ as the Killing form on $\mathfrak{sl}(1|1)$ is degenerate. Thus, the geometry associated to $\mathcal{H}(\mathfrak{sl}(1|1))$ is able to identify a "generalized Casimir" element of $\mathcal{U}(\mathfrak{sl}(1|1))$.

3. The Quantized Universal Enveloping Algebra of $\mathfrak{sl}(2,\Bbbk)$

3.1. The $\Bbbk$-Algebra $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$.

Let $\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk))$ denote the $\Bbbk$-algebra, known as quantized $\mathfrak{sl}(2,\Bbbk)$, defined as

$$\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk)) = \frac{\Bbbk\langle E,\ F,\ K,\ K^{-1}\rangle}{\left\langle\ KE - q^2EK,\ KF - q^{-2}FK,\ EF - FE - \frac{K^2-K^{-2}}{q^2-q^{-2}}\ \right\rangle},$$

where $q \in \Bbbk^\times$ and $q^4 \neq 1$ (cf. [18]).

It should be noted that this is not the current, official version of quantized $\mathfrak{sl}(2,\Bbbk)$; that version replaces the relation $EF - FE - \frac{K^2-K^{-2}}{q^2-q^{-2}} = 0$ with $EF - FE - \frac{K-K^{-1}}{q-q^{-1}} = 0$ in the defining relations (cf. [11]). We thank S. P. Smith of the University of Washington for bringing to our attention that if A denotes the graded algebra defined below in Definition 3.1 and if $\mathcal{O}_q$ denotes the current official version of quantized $\mathfrak{sl}(2,\Bbbk)$ (cf. [11]), then the ring of degree-zero elements in $A[(KT)^{-1}]$ is isomorphic to $\mathcal{O}_c$, where $c^2 = q$.

Definition 3.1. We define $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ to be the quadratic $\Bbbk$-algebra on degree-one generators E, F, K, T with defining relations

$$KT = TK, \qquad KE = q^2EK, \qquad KF = q^{-2}FK,$$
$$ET = q^2TE, \qquad FT = q^{-2}TF, \qquad EF - FE = \frac{K^2 - T^2}{q^2 - q^{-2}},$$

where $q \in \Bbbk^\times$ and $q^4 \neq 1$.

Note that $\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk)) \cong \mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))/\langle KT-1\rangle$. We will establish in Theorem 3.2 that $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is a quadratic quantum $\mathbb{P}^3$.

Theorem 3.2. *The $\Bbbk$-algebra $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is a quadratic quantum $\mathbb{P}^3$.*

Proof. Let A denote the $\Bbbk$-algebra on generators E, K, T with defining relations

$$KT = TK, \qquad EK = q^{-2}KE, \qquad ET = q^2TE.$$

The algebra $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is isomorphic to the Ore extension (cf. [9]) $A[F;\,\phi,\,\delta]$, where ϕ and δ are an automorphism and a left ϕ-derivation, respectively, of A defined by

$$\phi(E) = E, \quad \phi(K) = q^2K, \quad \phi(T) = q^{-2}T, \quad \delta(E) = \frac{T^2 - K^2}{q^2 - q^{-2}}, \quad \delta(K) = \delta(T) = 0.$$

By [15], since A is a skew polynomial ring, $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is Auslander-regular and satisfies the Cohen-Macaulay property. By the definition of the Cohen-Macaulay property ([14, Definition 5.8]), $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ has polynomial growth and is, hence, AS-regular. By [14, Theorem 4.8], $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is a domain and, thus, T is a normal regular element of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$. It follows that $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is a normal regular extension of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))/\langle T\rangle$ (in the language of [13]), which is an Ore extension of a skew polynomial ring, so $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is AS-regular of global dimension four (cf. [13, Theorem 2.6, Corollary 2.7] and the paragraph after [13, Definition 3.1.1]). Hence, $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is a quadratic quantum $\mathbb{P}^3$. ■

Note that there exists an automorphism φ on $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ defined by $\varphi : E \leftrightarrow F$ and $K \leftrightarrow T$.

3.2. The Quantum Space of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$.

Denote $x_1 = E$, $x_2 = F$, $x_3 = K$ and $x_4 = T$.

Theorem 3.3. *For every $q \in \Bbbk^\times$ with $q^4 \neq 1$, the point scheme, $\mathfrak{p}(q)$, of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is the union of a line, two conics and two points:*

(i) $\mathfrak{p}_1 = \mathcal{V}(x_3,\ x_4)$,
(ii) $\mathfrak{p}_2 = \mathcal{V}(x_3,\ q^4x_4^2 + (q^4-1)^2x_1x_2)$,
(iii) $\mathfrak{p}_3 = \mathcal{V}(x_4,\ q^4x_3^2 + (q^4-1)^2x_1x_2)$,
(iv) $\mathfrak{p}_4 = \mathcal{V}(x_1,\ x_2,\ x_3 + x_4)$, *and*
(v) $\mathfrak{p}_5 = \mathcal{V}(x_1,\ x_2,\ x_3 - x_4)$.

Proof. The point scheme of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is computed in the same manner as in Theorem 2.3. The polynomials that define the scheme are given in [3, §5.4.1]. The zero locus of these polynomials is $\bigcup_{i=1}^5 \mathfrak{p}_i$.

The Jacobian matrix, J_q, of $\mathfrak{p}(q)$ is given in [3, §5.4.3]. We examine the zero locus of the 2×2 minors of J_q to determine the multiplicity of the points in $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$, and the zero locus of the 3×3 minors of J_q to determine the multiplicity of the points in $\mathfrak{p}_4$ and $\mathfrak{p}_5$ ([10]). The only points on which these minors vanish are $e_1, e_2 \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3$.

The coordinate ring of the point scheme is $\Bbbk[x_1,\, x_2,\, x_3,\, x_4]/I$, where I is the ideal generated by the polynomials in [3, §5.4.1]. Consider the plane $P = \mathcal{V}(x_2 - x_3 - x_4)$. The points of intersection of $\mathfrak{p}$ and P are

$$\left(-\frac{q^4}{(q^4-1)^2},\ 1,\ 0,\ 1\right), \quad \left(-\frac{q^4}{(q^4-1)^2},\ 1,\ 1,\ 0\right), \quad (0,\ 0,\ 1,\ -1), \quad (1,\ 0,\ 0,\ 0).$$

By inverting $x_1 + \frac{q^4}{(q^4-1)^2}x_3 + \frac{q^4}{(q^4-1)^2}x_4$ in the coordinate ring of $\mathfrak{p}(q) \cap P$, we can determine the multiplicity of e_1. This yields a ring isomorphic to a polynomial ring on generators x, y with relations $x^2 = 0$, $xy = 0$, $y^2 = 0$, which has dimension three. Thus, e_1 is a multiple point of $\mathfrak{p}(q)$ only as a consequence of it being an intersection point of three irreducible components of $\mathfrak{p}(q)$; using the automorphism φ, we can conclude the same applies to e_2. ■

Corollary 3.4. *Let $A = \mathcal{H}_q(\mathfrak{sl}(2, \Bbbk))$.*

(i) *The points in $\mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$ on which the defining relations of $\mathcal{H}_q(\mathfrak{sl}(2, \Bbbk))$ vanish are of the form (p, p) if $p \in \mathfrak{p}_1 \cup \mathfrak{p}_4 \cup \mathfrak{p}_5$, and, if $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathfrak{p}_j$, for $j = 2, 3$, are of the form $\Big((\alpha_1, \alpha_2, \alpha_3, \alpha_4),\ (\alpha_1, q^{4(-1)^j}\alpha_2, q^{-2}\alpha_3, q^2\alpha_4) \Big)$.*

(ii) *There exists an automorphism $\sigma : \mathfrak{p} \to \mathfrak{p}$ which, on closed points, is defined by*

$$\sigma(p) = \sigma(p_1, p_2, p_3, p_4) = \begin{cases} p, & p \in \mathfrak{p}_1 \cup \mathfrak{p}_4 \cup \mathfrak{p}_5 \\ (p_1, q^{4(-1)^j}p_2, q^{-2}p_3, q^2p_4), & p \in \mathfrak{p}_j, \text{ for } j = 2, 3. \end{cases}$$

Proof. Part (i) is shown by computation. The existence of the map in (ii) follows from (i) and [13]. ■

The quantum Casimir element of $\mathcal{U}_q(\mathfrak{sl}(2, \Bbbk))$ is $\Omega_q = EF + FE + \left(\frac{q^4+1}{q^4-1}\right)\left(\frac{K^2+K^{-2}}{q^2-q^{-2}}\right)$ and its image in $\mathcal{H}_q(\mathfrak{sl}(2, \Bbbk))$ is $\Omega'_q = x_1x_2 + x_2x_1 + \left(\frac{q^4+1}{q^4-1}\right)\left(\frac{x_3^2+x_4^2}{q^2-q^{-2}}\right)$.

If $p \in \mathfrak{p}_2 \cup \mathfrak{p}_3$, then $\Omega'_q(p, \sigma(p)) = 0$. Moreover, Ω'_q is a central element of $\mathcal{H}_q(\mathfrak{sl}(2, \Bbbk))$; therefore, the geometric data has identified Ω_q as a distinguished (central) element of $\mathcal{U}_q(\mathfrak{sl}(2, \Bbbk))$.

Before describing the line variety of $\mathcal{H}_q(\mathfrak{sl}(2, \Bbbk))$, we advise the reader that (III) in Theorem 3.5 is a correction of the corresponding result in [3]. We thank the anonymous referee for noticing an issue with the original statement.

Theorem 3.5. *The reduced variety of the line scheme of $\mathcal{H}_q(\mathfrak{sl}(2, \Bbbk))$ is given by $\mathfrak{L}(q) = \bigcup_{i=1}^3 \mathfrak{L}_i$ where*

(I) $\mathfrak{L}_1 = \mathcal{V}(M_{13},\ M_{23},\ M_{34})$,

(II) $\mathfrak{L}_2 = \mathcal{V}(M_{14},\ M_{24},\ M_{34})$, *and*

(III) $\mathfrak{L}_3$ *is given by the zero locus of the five polynomials:*

$$\begin{aligned} &q^4M_{34}^2 + (q^4-1)^2M_{14}M_{24}, && q^4(M_{13}^2 - M_{14}^2)M_{34} + (q^4-1)^2M_{12}M_{13}M_{14}, \\ &q^4M_{34}^2 + (q^4-1)^2M_{13}M_{23}, && q^4(M_{24}^2 - M_{23}^2)M_{34} + (q^4-1)^2M_{12}M_{23}M_{24}, \end{aligned}$$

$$M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}.$$

Proof. For details on the construction of the line scheme from the defining relations of a quadratic quantum $\mathbb{P}^3$, the reader is referred to [4, 17].

The polynomials that define the line scheme are given in [3, §5.4.4]. The reduced variety given by those polynomials is $\mathfrak{L}(q)$. In the computation of the component $\mathfrak{L}_3$, once a Gröbner basis is found (using degree, reverse-lexicographic ordering) and then homogenized, the following seven polynomials are found:

$$h_1 = q^4 M_{34}^2 + (q^4-1)^2 M_{14}M_{24}, \qquad h_2 = -M_{12}M_{24}M_{34} + M_{13}M_{24}^2 + \tfrac{q^4}{(q^4-1)^2} M_{23}M_{34}^2,$$
$$h_3 = q^4 M_{34}^2 + (q^4-1)^2 M_{13}M_{23}, \qquad h_4 = -M_{12}M_{13}M_{34} + M_{13}^2 M_{24} + \tfrac{q^4}{(q^4-1)^2} M_{14}M_{34}^2,$$
$$h_5 = q^4(M_{24}^2 - M_{23}^2)M_{34} + (q^4-1)^2 M_{12}M_{23}M_{24}, \qquad h_6 = q^4(M_{13}^2 - M_{14}^2)M_{34} + (q^4-1)^2 M_{12}M_{13}M_{14},$$
$$P = M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}.$$

However, h_2 and h_4 belong to the ideal generated by the other five polynomials, since $h_2 = (M_{23}/(q^4-1)^2)h_1 - M_{24}P$ and $h_4 = (M_{14}/(q^4-1)^2)h_3 - M_{13}P$. For additional details of the computation, the reader is referred to [3]. ■

The next result describes the lines in $\mathbb{P}^3$ that are parametrized by the closed points of the line scheme of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$. We thank S. P. Smith of the University of Washington for his suggestion to consider a pencil of quadrics in $\mathbb{P}^3$.

Corollary 3.6. *Let $\mathfrak{L}(q) = \bigcup_{i=1}^3 \mathfrak{L}_i$ be as above. The lines in $\mathbb{P}^3$ that correspond to the line modules of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ are precisely the lines in the pencil of quadrics*

$$Q_q(\alpha,\,\beta) = \mathcal{V}\left(\ \alpha q^4(x_3^2 + x_4^2) + \alpha(q^4-1)^2 x_1x_2 + \beta q^4 x_3x_4\ \right),$$

where $(\alpha,\beta) \in \mathbb{P}^1$. More precisely,

(i) *$\mathfrak{L}_1$ gives all lines in $\mathcal{V}(x_3)$,*
(ii) *$\mathfrak{L}_2$ gives all lines in $\mathcal{V}(x_4)$, and*
(iii) *$\mathfrak{L}_3$ gives the union of the following three families of lines:*
  (a) *the lines in $\mathcal{V}(x_2)$ that pass through e_1,*
  (b) *the lines in $\mathcal{V}(x_1)$ that pass through e_2, and*
  (c) *the lines in $\mathbb{P}^3$ of the form $\mathcal{V}(x_1 - a_1x_3 - b_1x_4,\ x_2 - a_2x_3 - b_2x_4)$ for all $a_1, a_2, b_1, b_2 \in \Bbbk$ such that $q^4 + (q^4-1)^2 a_1a_2 = 0$ and $a_1a_2 = b_1b_2$.*

Proof. Let $(a_1,\,a_2,\,a_3,\,a_4)$, $(b_1,\,b_2,\,b_3,\,b_4) \in \mathbb{P}^3$ be distinct points and let

$$\ell = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

represent the line through them.

(i) If ℓ is given by $\mathfrak{L}_1$, then it follows that ℓ is represented by

$$\begin{bmatrix} a_1 & a_2 & 0 & a_4 \\ b_1 & b_2 & 0 & b_4 \end{bmatrix}.$$

Thus, ℓ belongs to $\mathcal{V}(x_3)$ if and only if ℓ is given by $\mathfrak{L}_1$.

(ii) Applying the automorphism φ to the lines described by $\mathfrak{L}_1$ gives that $\mathfrak{L}_2$ describes all lines in $\mathcal{V}(x_4)$.

(iii) Assume that ℓ is given by $\mathfrak{L}_3 \setminus (\mathfrak{L}_1 \cup \mathfrak{L}_2)$.

If $M_{34} = 0$, then we may assume that

$$\ell = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_5 & c_6 & 0 & 0 \end{bmatrix},$$

where $c_1, \ldots, c_6 \in \Bbbk$ satisfy the polynomials that define $\mathfrak{L}_3$. In particular, $c_5 c_6 = 0$. If $c_6 \neq 0$, then ℓ may be represented by

$$\begin{bmatrix} c_1 & 0 & c_3 & c_4 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

where $c_1 c_3 c_4 = 0$. The cases where $c_3 = 0$ or $c_4 = 0$ are described by $\mathfrak{L}_1$ and $\mathfrak{L}_2$. If $c_1 = 0$, then

$$\ell = \begin{bmatrix} 0 & 0 & c_3 & c_4 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

which belongs to $\mathcal{V}(x_1)$ and passes through e_2. By a similar argument, if $c_5 \neq 0$, then ℓ belongs to $\mathcal{V}(x_2)$ and passes through e_1.

If $M_{34} \neq 0$, we may take $M_{34} = 1$ and write

$$\ell = \begin{bmatrix} a_1 & a_2 & 1 & 0 \\ b_1 & b_2 & 0 & 1 \end{bmatrix},$$

where $q^4 + (q^4 - 1)^2 a_1 a_2 = 0$ and $a_1 a_2 = b_1 b_2$. Any point on a line with this representation belongs to $\mathcal{V}(x_1 - a_1 x_3 - b_1 x_4,\ x_2 - a_2 x_3 - b_2 x_4)$.

The lines given by $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are precisely those on $Q_q(0,\ 1)$. The lines given by $\mathfrak{L}_3$ of the form

$$\begin{bmatrix} 0 & 0 & c_3 & c_4 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & c_3 & c_4 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

for $c_3 c_4 \neq 0$, belong to $Q_q\left(1,\ -\frac{c_3^2 + c_4^2}{c_3 c_4}\right)$. If $\ell = \mathcal{V}(x_1 - a_1 x_3 - b_1 x_4,\ x_2 - a_2 x_3 - b_2 x_4)$, where $a_1, a_2, b_1, b_2 \in \Bbbk$, $q^4 + (q^4 - 1)^2 a_1 a_2 = 0$ and $a_1 a_2 = b_1 b_2$, then $\ell \in Q_q\left(1,\ -\frac{(q^4-1)^2}{q^4}(a_1 b_2 + a_2 b_1)\right)$.

If $\alpha \neq 0$ and $\beta^2 \neq 4$, then $Q_q(1,\ \beta)$ has rulings

$$\left\{\mathcal{V}\left(\ x_1 - \mu q^4 (x_3 - \delta_1 x_4),\ \ \mu(q^4 - 1)^2 x_2 + x_3 - \delta_2 x_4\ \right) : \mu \in \mathbb{P}^1\right\}$$

and

$$\left\{\mathcal{V}\left(\ x_2 - \mu q^4 (x_3 - \delta_1 x_4),\ \ \mu(q^4 - 1)^2 x_1 + x_3 - \delta_2 x_4\ \right) : \mu \in \mathbb{P}^1\right\},$$

where δ_1 and δ_2 are distinct solutions of $\delta^2 + \beta\delta + 1 = 0$.

If $\alpha \neq 0$ and $\beta^2 = 4$, then

$$Q_q(1,\ \beta) = \mathcal{V}\left(\ q^4 \left(x_3 + \frac{\beta}{2} x_4\right)^2 + (q^4 - 1)^2 x_1 x_2\ \right)$$

which is a rank-three quadric and so has only one ruling, namely

$$\left\{\mathcal{V}\left(\ x_1-\mu q^4\left(x_3+\frac{\beta}{2}x_4\right),\ \ \mu(q^4-1)^2x_2+x_3+\frac{\beta}{2}x_4\ \right):\mu\in\mathbb{P}^1\right\}.$$

The lines in each of these rulings are given by $\mathfrak{L}_3$. The result follows. ■

4. The Color Lie Algebra $\mathfrak{sl}_k(2,\Bbbk)$

4.1. The $\Bbbk$-Algebra $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$.

Let $\{e,\ f,\ h\}$ be the standard basis of $\mathfrak{sl}(2,\Bbbk)$ and define

$$a_1=\frac{i}{2}(e-f),\qquad a_2=-\frac{1}{2}(e+f),\qquad a_3=\frac{i}{2}h,$$

so that the bracket on $\mathfrak{sl}(2,\Bbbk)$ is defined by

$$[a_1,a_2]=-a_3,\qquad [a_2,a_3]=a_1,\qquad [a_1,a_3]=-a_2.$$

Let $G=\mathbb{Z}_2\times\mathbb{Z}_2$ and define a G-grading on $\mathfrak{sl}(2,\Bbbk)$ by $\mathfrak{sl}(2,\Bbbk)=\bigoplus_{g\in G}X_g$, where

$$X_0=\{0\},\qquad X_{(1,0)}=\Bbbk a_1,\qquad X_{(0,1)}=\Bbbk a_2,\qquad X_{(1,1)}=\Bbbk a_3.$$

Define a bicharacter map $\epsilon:G\times G\to\Bbbk^\times$ by $\epsilon\left((\alpha_1,\ \alpha_2),\ (\beta_1,\ \beta_2)\right)=(-1)^{\alpha_1\beta_2-\alpha_2\beta_1}$.

The color Lie algebra $\mathfrak{sl}_k(2,\Bbbk)$, called Klein $\mathfrak{sl}(2,\Bbbk)$, is the ϵ-Lie algebra with bracket

$$[a_1,\ a_2]=a_3,\qquad [a_2,\ a_3]=a_1,\qquad [a_3,\ a_1]=a_2.$$

For more details on the construction of $\mathfrak{sl}_k(2,\Bbbk)$ from $\mathfrak{sl}(2,\Bbbk)$, the reader is referred to [5].

The universal enveloping algebra of $\mathfrak{sl}_k(2,\Bbbk)$ is

$$\mathcal{U}(\mathfrak{sl}_k(2,\Bbbk))=\frac{\Bbbk\langle a_1,\ a_2,\ a_3\rangle}{\langle\ a_1a_2+a_2a_1-a_3,\ \ a_2a_3+a_3a_2-a_1,\ \ a_3a_1+a_1a_3-a_2\ \rangle}.$$

Definition 4.1. The homogenization, $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$, of $\mathcal{U}(\mathfrak{sl}_k(2,\Bbbk))$ is the $\Bbbk$-algebra on degree-one generators a_1, a_2, a_3, a_4 with defining relations

$$a_1a_2+a_2a_1=a_3a_4,\qquad a_2a_3+a_3a_2=a_1a_4,\qquad a_3a_1+a_1a_3=a_2a_4,$$
$$a_1a_4=a_4a_1,\qquad a_2a_4=a_4a_2,\qquad a_3a_4=a_4a_3.$$

Note that $\mathcal{U}(\mathfrak{sl}_k(2,\Bbbk))\cong\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))/\langle a_4-1\rangle$. Unlike $\mathcal{H}(\mathfrak{sl}(1|1))$ and $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$, it is not clear if $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$ is an Ore extension of an AS-regular algebra of global dimension three. We show regularity, instead, in Theorem 4.2, via a result of [13].

Theorem 4.2. *The $\Bbbk$-algebra $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$ is a quadratic quantum $\mathbb{P}^3$.*

Proof. A computation using Bergman's Diamond Lemma shows that a $\Bbbk$-basis for $\mathcal{U}(\mathfrak{sl}_k(2, \Bbbk))$ is $\mathfrak{B} = \{a_1^i a_2^j a_3^k : i, j, k \in \mathbb{N}\}$. Let $D = \mathcal{H}(\mathfrak{sl}_k(2, \Bbbk))$. Note that $A = D/\langle a_4 \rangle$ is a skew-polynomial ring on three variables and therefore is an AS-regular algebra. We will prove that D is a central regular extension of A in the sense of [13].

Suppose $a_4 f = 0$ in D for some $f \in D \setminus \Bbbk$ and view $a_1 < a_2 < a_3 < a_4$. We may assume $f = g_1 + \cdots + g_m \in D$ is homogeneous, where $g_1, \ldots, g_m$ are linearly independent scalar multiples of monomials with the generators of D in increasing order. Let $\bar{f}$ denote the image of f in $D/\langle a_4 - 1 \rangle$. Since $a_4 f = 0$ in D, we have $\bar{f} = 0$ in $D/\langle a_4 - 1 \rangle \cong \mathcal{U}(\mathfrak{sl}_k(2, \Bbbk))$. Since the g_i are written with the generators in increasing order, it follows that an element of the form $a_1^i a_2^j a_3^k$, for some $i, j, k \in \mathbb{N}$, is missing from $\mathfrak{B}$, which is a contradiction. Hence, a_4 is regular in D.

It follows that D is a central regular extension of A and, by [13], is therefore an AS-regular algebra of global dimension four (cf. [13, Theorem 2.6, Corollary 2.7] and the paragraph after [13, Definition 3.1.1]). Moreover, A is a skew-polynomial ring, so it is Auslander-regular. Hence, by [14, §5.10], D is also Auslander-regular and satisfies the Cohen-Macaulay property. ■

We note that there is an automorphism φ on $\mathcal{H}(\mathfrak{sl}_k(2, \Bbbk))$ of order three given by $a_1 \mapsto a_2 \mapsto a_3 \mapsto a_1$ and $a_4 \mapsto a_4$.

4.2. **The Quantum Space of $\mathcal{H}(\mathfrak{sl}_k(2, \Bbbk))$.**

Recall the notation $e_1, \ldots, e_4$ from Section 1.

Theorem 4.3. *The point scheme, $\mathfrak{p}$, of $\mathcal{H}(\mathfrak{sl}_k(2, \Bbbk))$ is the union of three lines and five points*:

(i) $\mathfrak{p}_1 = \mathcal{V}(a_1, a_4)$,
(ii) $\mathfrak{p}_2 = \mathcal{V}(a_2, a_4)$,
(iii) $\mathfrak{p}_3 = \mathcal{V}(a_3, a_4)$,
(iv) $\mathfrak{p}_4 = \mathcal{V}(a_1, a_2, a_3)$,
(v) $\mathfrak{p}_5 = \mathcal{V}(a_2 + a_1, a_3 + a_1, a_4 - 2a_1)$,
(vi) $\mathfrak{p}_6 = \mathcal{V}(a_2 + a_1, a_3 - a_1, a_4 + 2a_1)$,
(vii) $\mathfrak{p}_7 = \mathcal{V}(a_2 - a_1, a_3 - a_1, a_4 - 2a_1)$,
(viii) $\mathfrak{p}_8 = \mathcal{V}(a_2 - a_1, a_3 + a_1, a_4 + 2a_1)$,

where the points $e_1 \in \mathfrak{p}_2 \cap \mathfrak{p}_3$, $e_2 \in \mathfrak{p}_1 \cap \mathfrak{p}_3$, $e_3 \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ are counted with multiplicity three and all other points are reduced.

Proof. The polynomials that define $\mathfrak{p}$ are listed in [3, §5.3.1]. The zero locus of these polynomials is $\bigcup_{i=1}^{8} \mathfrak{p}_i$.

The Jacobian matrix, J, of the point scheme is given in [3, §5.3.3]. We examine the zero locus of the 2×2 minors of J to determine the multiplicity of the points in $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$, and the zero locus of the 3×3 minors of J to determine the multiplicity of the points in $\mathfrak{p}_4, \ldots, \mathfrak{p}_8$ ([10]). The only points where these minors vanish are the points e_1, e_2 and e_3.

The coordinate ring of the point scheme is $\Bbbk[a_1, a_2, a_3, a_4]/I$, where I is the ideal generated by the polynomials in [3, §5.3.1]. Consider the plane $P = \mathcal{V}(a_2 - a_3 - a_4)$. The points of intersection of $\mathfrak{p}$ and P are e_1 and $(0, 1, 1, 0)$. We localize the coordinate ring of $\mathfrak{p} \cap P$ around e_1 and obtain an algebra that is isomorphic to a polynomial ring in one variable, x, with exactly one relation, $x^3 = 0$, and so has dimension three. Therefore, e_1 has multiplicity three; by using the automorphisms φ and φ^2, it follows that e_2 and e_3 also have multiplicity three. ■

Corollary 4.4. *Let* $A = \mathcal{H}(\mathfrak{sl}_k(2, \Bbbk))$.

(i) *The points in* $\mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$ *on which the defining relations of* $\mathcal{H}(\mathfrak{sl}_k(2, \Bbbk))$ *vanish are of the form* (p, p) *if* $p \in \bigcup_{i=4}^{8} \mathfrak{p}_i$, *and are of the form*

$$\big((\alpha_1, \alpha_2, \alpha_3, 0), \quad ((-1)^{\delta_{j2}}\alpha_1, (-1)^{\delta_{j3}}\alpha_2, (-1)^{\delta_{j1}}\alpha_3, 0) \big)$$

if $(\alpha_1, \alpha_2, \alpha_3, 0) \in \mathfrak{p}_j$, *for* $j = 1, 2, 3$, *where* δ_{jk} *is the Kronecker-delta symbol.*

(ii) *There exists an automorphism* $\sigma : \mathfrak{p} \to \mathfrak{p}$ *which, on closed points, is defined by*

$$\sigma(p) = \sigma(p_1, p_2, p_3, p_4) = \begin{cases} p, & p \in \mathfrak{p}_j, j = 4, \ldots, 8 \\ ((-1)^{\delta_{j2}}p_1, (-1)^{\delta_{j3}}p_2, (-1)^{\delta_{j1}}p_3, 0), & p \in \mathfrak{p}_j, j = 1, 2, 3. \end{cases}$$

Proof. Part (i) is shown by computation. The existence of the map in (ii) follows from (i) and [13]. ■

Theorem 4.5. *The line scheme of* $\mathcal{H}(\mathfrak{sl}_k(2, \Bbbk))$ *has dimension two and its reduced variety* $\mathfrak{L}$ *is the union of a plane and twelve lines*:

(I) $\mathfrak{L}_0 = \mathcal{V}(M_{14},\ M_{24},\ M_{34})$,
(II) $\mathfrak{L}_1 = \mathcal{V}(M_{12},\ M_{34},\ M_{14} - M_{24},\ M_{13} - M_{23})$,
(III) $\mathfrak{L}_2 = \mathcal{V}(M_{12},\ M_{34},\ M_{14} + M_{24},\ M_{13} + M_{23})$,
(IV) $\mathfrak{L}_3 = \mathcal{V}(M_{34},\ M_{14} + 2M_{23},\ 2M_{13} + M_{24},\ 2M_{23} - M_{24})$,
(V) $\mathfrak{L}_4 = \mathcal{V}(M_{34},\ M_{14} + 2M_{23},\ 2M_{13} + M_{24},\ 2M_{23} + M_{24})$,
(VI) $\mathfrak{L}_5 = \mathcal{V}(M_{13},\ M_{24},\ M_{34} - M_{14},\ M_{12} + M_{23})$,
(VII) $\mathfrak{L}_6 = \mathcal{V}(M_{13},\ M_{24},\ M_{34} + M_{14},\ M_{12} - M_{23})$,
(VIII) $\mathfrak{L}_7 = \mathcal{V}(M_{24},\ M_{34} + 2M_{12},\ M_{14} - 2M_{23},\ 2M_{12} - M_{14})$,
(IX) $\mathfrak{L}_8 = \mathcal{V}(M_{24},\ M_{34} + 2M_{12},\ M_{14} - 2M_{23},\ 2M_{12} + M_{14})$,
(X) $\mathfrak{L}_9 = \mathcal{V}(M_{14},\ M_{23},\ M_{24} - M_{34},\ M_{13} - M_{12})$,
(XI) $\mathfrak{L}_{10} = \mathcal{V}(M_{14},\ M_{23},\ M_{24} + M_{34},\ M_{13} + M_{12})$,
(XII) $\mathfrak{L}_{11} = \mathcal{V}(M_{14},\ M_{24} - 2M_{13},\ M_{34} - 2M_{12},\ M_{34} + 2M_{13})$,
(XIII) $\mathfrak{L}_{12} = \mathcal{V}(M_{14},\ M_{24} - 2M_{13},\ M_{34} - 2M_{12},\ M_{34} - 2M_{13})$.

Proof. For details on the construction of the line scheme from the defining relations of a quadratic quantum $\mathbb{P}^3$, the reader is referred to [4, 17].

The polynomials that define the line scheme are given in [3, §5.3.4]. The reduced variety given by those polynomials is $\mathfrak{L}$. For details of the computation, the reader is referred to [3]. ■

Corollary 4.6. *The lines in* $\mathbb{P}^3$ *that are parametrized by the line scheme of* $\mathcal{H}(\mathfrak{sl}_k(2, \Bbbk))$ *are:*

(i) *all lines in* $\mathcal{V}(a_4)$*,*
(ii) *those lines in* $\mathcal{V}(a_1 - a_2)$ *that pass through* $(1, 1, 0, 0)$*,*
(iii) *those lines in* $\mathcal{V}(a_1 + a_2)$ *that pass through* $(1, -1, 0, 0)$*,*
(iv) *those lines in* $\mathcal{V}(a_4 - 2a_3)$ *that pass through* $(1, -1, 0, 0)$*,*
(v) *those lines in* $\mathcal{V}(a_4 + 2a_3)$ *that pass through* $(1, 1, 0, 0)$*,*
(vi) *those lines in* $\mathcal{V}(a_1 - a_3)$ *that pass through* $(1, 0, 1, 0)$*,*
(vii) *those lines in* $\mathcal{V}(a_1 + a_3)$ *that pass through* $(1, 0, -1, 0)$*,*
(viii) *those lines in* $\mathcal{V}(a_4 - 2a_2)$ *that pass through* $(1, 0, -1, 0)$*,*
(ix) *those lines in* $\mathcal{V}(a_4 + 2a_2)$ *that pass through* $(1, 0, 1, 0)$*,*
(x) *those lines in* $\mathcal{V}(a_2 - a_3)$ *that pass through* $(0, 1, 1, 0)$*,*
(xi) *those lines in* $\mathcal{V}(a_2 + a_3)$ *that pass through* $(0, 1, -1, 0)$*,*
(xii) *those lines in* $\mathcal{V}(a_4 - 2a_1)$ *that pass through* $(0, 1, -1, 0)$*, and*
(xiii) *those lines in* $\mathcal{V}(a_4 + 2a_1)$ *that pass through* $(0, 1, 1, 0)$*.*

Proof. Let $(b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \in \mathbb{P}^3$ be distinct points and let

$$\ell = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

represent the line through them.

(i) If ℓ is given by $\mathfrak{L}_0$, then it follows that ℓ is given by

$$\begin{bmatrix} b_1 & b_2 & b_3 & 0 \\ c_1 & c_2 & c_3 & 0 \end{bmatrix}.$$

So, ℓ is given by $\mathfrak{L}_0$ if and only if ℓ belongs to $\mathcal{V}(a_4)$.

(ii) If ℓ is given by $\mathfrak{L}_1$, then it follows that ℓ may be represented as

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & d_1 & d_2 \end{bmatrix},$$

where $(d_1, d_2) \in \mathbb{P}^1$. Therefore, ℓ passes through $(1, 1, 0, 0)$ and belongs to the plane $\mathcal{V}(a_1 - a_2)$.

(iii) The result follows by an argument similar to that in (ii).

(iv) If ℓ is given by $\mathfrak{L}_3$, then

$$\ell = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 \\ d_5 & d_6 & 0 & 0 \end{bmatrix},$$

where $d_1, \ldots, d_6 \in \Bbbk$ satisfy the polynomials that define $\mathfrak{L}_3$.

If $d_6 = 0$, then it follows that ℓ may be represented as

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

which is the line $\mathcal{V}(a_3,\, a_4)$ (this line is also given by $\mathfrak{L}_0$).

If $d_6 \neq 0$, then it follows that $\ell = \mathcal{V}(a_3,\, a_4)$ or

$$\ell = \begin{bmatrix} 0 & d_7 & d_3 & 2d_3 \\ 1 & -1 & 0 & 0 \end{bmatrix},$$

where $d_7 \in \mathbb{k}$. Therefore, $\mathfrak{L}_3$ gives all lines that pass through $(1, -1, 0, 0)$ and belong to $\mathcal{V}(a_4 - 2a_3)$.

(v) The result follows by an argument similar to that in (iv).

To obtain the lines given by $\mathfrak{L}_5, \ldots, \mathfrak{L}_8$, one may apply φ^2 to the lines described by $\mathfrak{L}_1, \ldots, \mathfrak{L}_4$, respectively. To obtain the lines given by $\mathfrak{L}_9, \ldots, \mathfrak{L}_{12}$, one may apply φ to the lines described by $\mathfrak{L}_1, \ldots, \mathfrak{L}_4$, respectively. ■

We conclude by remarking that the quantum space of $\mathcal{H}(\mathfrak{sl}_k(2, \mathbb{k}))$ does not appear to identify any distinguished element of $\mathcal{H}(\mathfrak{sl}_k(2, \mathbb{k}))$. This behavior is in stark contrast to that of $\mathcal{H}(\mathfrak{sl}(2, \mathbb{k}))$, $\mathcal{H}(\mathfrak{sl}(1|1))$ and $\mathcal{H}_q(\mathfrak{sl}(2, \mathbb{k}))$, where the quantum space in each case identified distinguished elements of the algebra, including an analogue of a Casimir element for each associated enveloping algebra.

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References

[1] M. Artin, J. Tate and M. Van den Bergh, Some Algebras Associated to Automorphisms of Elliptic Curves, *in* "The Grothendieck Festschrift **1**", pp 33-85, Eds. P. Cartier et al, Birkhäuser Boston (1990).

[2] M. Artin, J. Tate and M. Van den Bergh, Modules Over Regular Algebras of Global Dimension 3, *Invent. Math.* **106** (1991), 333-388.

[3] R. Chandler, On the Quantum Spaces of some Quadratic Regular Algebras of Global Dimension Four, Ph.D. Thesis, Univ. of Texas at Arlington (2016). https://rc.library.uta.edu/uta-ir/handle/10106/25898.

[4] R. Chandler and M. Vancliff, The One-Dimensional Line Scheme of a Certain Family of Quantum $\mathbb{P}^3$s, *J. Alg.* **439** (2015), 316-333.

[5] X. W. Chen, S. D. Silvestrov, and F. Van Oystaeyen, Representations and Cocycle Twists of Color Lie Algebras, *Springer Sci.-Alg. Rep. Theory* **9** (2006), 633-650.

[6] S. Cheng and W. Wang, *Dualities and Representations of Lie Superalgebras*, Graduate Studies in Math. **144**, Amer. Math. Soc., Rhode Island, 2013.

[7] D. A. Cox, J. B. Little, and D. O'Shea, *Ideals, Varieties and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Springer-Verlag, New York, 1992.

[8] L. D. Faddeev, N. Y. Reshetikhin, and L. A. Takhtadzhyan, Quantization of Lie Groups and Lie Algebras, *Leningrad Math. J.* **1** (1990), 193-225.

[9] K. R. Goodearl and R. B. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings* (*2nd Ed.*), Lond. Math. Soc. Student Texts **61**, Cambridge Univ. Press, 2004.

[10] J. Harris, *Algebraic Geometry: A First Course*, Graduate Texts in Math. **133**, Springer-Verlag, New York, 1992.

[11] J. C. Jantzen, *Lectures on Quantum Groups*, Graduate Studies in Math. **6**, Amer. Math. Soc., Rhode Island, 1996.

[12] L. Le Bruyn and S. P. Smith, Homogenized $\mathfrak{sl}(2)$, *Proc. Amer. Math. Soc.* **118** No. 3 (1993), 725-730.

[13] L. Le Bruyn, S. P. Smith and M. Van den Bergh, Central Extensions of Three-Dimensional Artin-Schelter Regular Algebras, *Math. Z.* **222** (1996), 171-212.

[14] T. Levasseur, Some Properties of Noncommutative Regular Graded Rings, *Glasgow Math. J.* **34** (1992), 277-300.

[15] T. Levasseur and T. Stafford, The Quantum Coordinate Ring of the Special Linear Group, *J. Pure and Appl. Alg.* **86** (1993), 181-186.

[16] B. Shelton and M. Vancliff, Schemes of Line Modules I, *J. Lond. Math. Soc.* **65** No. 3 (2002), 575-590.

[17] B. Shelton and M. Vancliff, Schemes of Line Modules II, *Comm. Alg.* **30** No. 5 (2002), 2535-2552.

[18] S. P. Smith, Quantum Groups: An Introduction and Survey for Ring Theorists, *in* "Noncommutative Rings," *MSRI Publications* **24** (1992), 131-178.

[19] M. Vancliff, Quadratic Algebras Associated with the Union of a Quadric and a Line in $\mathbb{P}^3$, *J. Alg.* **165** (1994), 63-90.