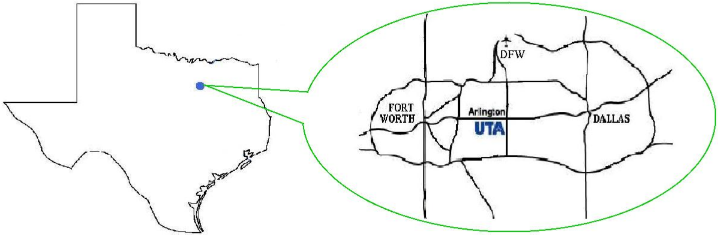
Defining a Notion of Non-commutative Complete Intersection via Base-Point Modules

Michaela Vancliff

UNIVERSITY OF TEXAS ARLINGTON

http://www.uta.edu/math/vancliff/R/ci.pdf   vancliff@uta.edu



NSF grants DMS-0900239 & DMS-1302050

Main Goal

... generalize the following well-known commutative result to certain non-commutative algebras satisfying good homological properties:

Theorem ([standard])

Let R denote the (commutative) polynomial ring on n variables over an algebraically closed field $\Bbbk$. Homogeneous elements $f_1, \ldots, f_m \in R_{>0}$ define/yield a complete intersection if one of the following equivalent conditions holds:

(a) *$\{f_1, \ldots, f_m\}$ is a regular sequence in R (i.e., for all i, f_i is not a zero divisor in $R/\langle f_1, \ldots, f_{i-1}\rangle$);*

(b) *for all $k = 1, \ldots, m$, $GKdim(R/\langle f_1, \ldots, f_k\rangle) = n - k$.*

Moreover, if $m = n$, then there exist another 2 equiv conditions equiv to (a) and (b):

(c) $\dim_{\Bbbk}(R/\langle f_1, \ldots, f_n\rangle) < \infty$;

(d) $\mathcal{V}(f_1, \ldots, f_n) \subset \mathbb{P}^{n-1}$ *is empty.*

Assumptions & Notation

- $\mathbb{k}$ = algebraically closed field.
- $A = \bigoplus_{i=0}^{\infty} A_i$ = an $\mathbb{N}$-graded, connected ($A_0 = \mathbb{k}$), finitely generated $\mathbb{k}$-algebra, generated by A_1.
- $\text{GKdim}(A) = n \in \mathbb{N}$.
- A has a normalizing sequence $F = \{f_1, \ldots, f_m\} \subset A \setminus \mathbb{k}^{\times}$ consisting of homogeneous elements (i.e., for all i, f_i is normal in $A/\langle f_1, \ldots, f_{i-1}\rangle$).

The Desired Equivalent Conditions

In order to address the notion of complete intersection in the context of A and F, we will be interested in 4 (possibly equivalent) conditions on A and F as follows:

I. F is a regular sequence in A;

II. for each $k = 1, \ldots, m$, $\mathrm{GKdim}(A/\langle f_1, \ldots, f_k \rangle) = n - k$;
(last slide: $\mathrm{GKdim}(A) = n$)

III. $\dim_{\Bbbk}(A/\langle F \rangle) < \infty$;

IV. $\widehat{\mathcal{V}(F)}$ is empty (to be defined below).

Base-Point Modules

In past work with T. Cassidy on skew poly rings S, he & I initially thought $\widehat{\mathcal{V}(F)}$ should be the scheme of point modules over $S/\langle F\rangle$. However, J. T. Stafford led us to realise that, even for skew poly rings, one needs to consider **both** point modules **&** fat point modules. The same is true for the algebra A in this talk....

Definition ([CV2])

A right **base-point module** over A is any 1-critical (wrt GK-dimension) graded right A-module that is generated by its homogeneous degree-zero elements & has Hilbert series $H(t) = c/(1-t)$ for some $c \in \mathbb{N}$.

So, $c = 1 \Leftrightarrow$ point module, and $c \geq 2 \Leftrightarrow$ fat point module.

Notation: $\widehat{\mathcal{V}(F)}$ is the collection of (right) base-point modules over $A/\langle F\rangle$.

Example (See [ATV1], [CV2])

If $A =$ poly ring & $\deg(f_i) = 2$ for all i, then F is a quadric system & there is a one-to-one corresp between $\widehat{\mathcal{V}(F)}$ and the base points of the quadric system.

Proposed Definition

Definition

Let A be as above and suppose conditions I-IV are equivalent for all normalizing sequences F in A of the above form that have length n. For such an F, if the equivalent conditions I-IV hold, call $A/\langle F\rangle$ a **complete intersection**.

Are there many algebras A to which this definition applies?

Examples

- Regular skew poly rings, i.e., $S = \mathbb{k}$-algebra on generators $z_1, \ldots, z_n$ where $z_j z_i = \mu_{ij} z_i z_j$ for all $i \neq j$ such that $\mu_{ij} \in \mathbb{k}^\times$ & $\mu_{ij}\mu_{ji} = 1$.
- Regular graded Clifford algebras (with $\mathrm{char}(\mathbb{k}) \neq 2$).
- Homogenization of the universal enveloping algebra of any Lie algebra of dimension $n - 1 < \infty$.

Main Theorem

Initial generalization to non-commutative setting was to the case of skew poly rings with T. Cassidy in [CV1] & [CV2].

Later, I generalized it further as follows:

Theorem

Let $A = \bigoplus_{i=0}^{\infty} A_i$ *denote a connected,* $\mathbb{N}$*-graded* $\mathbb{k}$*-algebra that is generated by* A_1*. Suppose that A is (noetherian) Auslander-Gorenstein of finite injective dimension & satisfies the Cohen-Macaulay property, & that there exists a normalizing sequence* $\{y_1, \ldots, y_\nu\} \subset A \setminus \mathbb{k}^\times$ $(\ \nu < \infty\ )$ *consisting of homog elements such that* $GKdim(A/\langle y_1, \ldots, y_\nu \rangle) = 1$*.*
If $GKdim(A) = n \in \mathbb{N}$*, then conditions I-IV are equivalent for any normalizing sequence* $F \subset A \setminus \mathbb{k}^\times$ *that consists of n homogeneous elements.*

Most of the result is straightforward to prove & follows from work in [CV1]; the part that needs work is III $\Leftrightarrow$ IV.

Next few slides = outline of proof of III $\Leftrightarrow$ IV.

Recall

III. $\dim_{\Bbbk}(A/\langle F \rangle) < \infty$;

IV. $\widehat{\mathcal{V}(F)}$ is empty.

Outline of Proof of III $\Leftrightarrow$ IV

III. $\dim_{\Bbbk}(A/\langle F\rangle) < \infty$; IV. $\widehat{\mathcal{V}(F)} = \emptyset$

$\text{IV}' \Rightarrow \text{III}'$

Suppose $M = \bigoplus_{i=0}^{\infty} M_i \in \widehat{\mathcal{V}(F)}$. So $M = M_0(A/\langle F\rangle)$ and $\dim_{\Bbbk}(M_0) < \infty$, while $\dim_{\Bbbk}(M) = \infty$; thus, $\dim_{\Bbbk}(A/\langle F\rangle) = \infty$.

$\text{III}' \Rightarrow \text{IV}'$

Let $I = \langle F\rangle$ & suppose $\dim_{\Bbbk}(A/I) = \infty$. Setting $y_0 = 0 \in A$, $\exists\ \theta \in \{0, \ldots, \nu\}$ such that $\text{GKdim}(A/(I + \sum_{i=0}^{\theta} y_i A)) = 1$.

Let k denote the smallest such θ, & let $\mathcal{A} = A/(I + \sum_{i=0}^{k} y_i A)$, which is a $\Bbbk$-algebra since $\{y_1, \ldots, y_\nu\}$ is a normalizing sequence in A. By construction, $\text{GKdim}(\mathcal{A}) = 1$.

We may now apply Example 5.5 of [AZ] to $\mathcal{A}$ as follows.

III$'$ $\Rightarrow$ IV$'$ (cont'd)

By [SSW], $\mathcal{A}$ is P.I. & finitely generated over its center & contains a regular homogeneous central element z of positive degree $d \in \mathbb{N}$.

Let $B = \mathcal{A}[z^{-1}]$ and $R = B_0$. The algebras $\mathcal{A}$ and B are locally finite, so $\dim_{\Bbbk}(R) < \infty$ & so R has a finite-dimensional simple module N.

Goal: build a BPM over $\mathcal{A}$ from N.

Corollary I.3.26 in [NV] $\Rightarrow B \cong R[x, x^{-1}; \sigma]$ for some $x \in B_1$, $\sigma \in \mathrm{Aut}(R)$.

$\text{III}' \Rightarrow \text{IV}'$ (cont'd) $\quad (B = \mathcal{A}[z^{-1}],\ R = B_0,\ B \cong R[x, x^{-1}; \sigma])$

Still following Example 5.5 of [AZ], we can now apply Example 5.4 of [AZ], to obtain that spec$(R) \cong$ proj$(\mathcal{A})$, where proj$(\mathcal{A})$ is the category of fin gen graded $\mathcal{A}$-modules mod the subcat of fin gen graded torsion $\mathcal{A}$-modules.

In particular, by §7 of [ATV2], we set

$\hat{N} = \bigoplus_{i=0}^{\infty}(N \otimes_R B)_i = \bigoplus_{i=0}^{\infty}(N \otimes_R B_i)$, which is a fin gen graded $\mathcal{A}$-module, generated by $\hat{N}_0 = N \otimes \Bbbk$. As vector spaces, $\hat{N}_i = N \otimes \Bbbk x^i \quad \forall\ i$.

Goal: prove $\hat{N}$ is a BPM over $\mathcal{A}$. $\quad (\Rightarrow \hat{N} \in \widehat{\mathcal{V}(F)}$ & so IV$'$ holds.)

It suffices to prove $\hat{N}$ is 1-critical. Standard method uses x normal, z acts faithfully on $\hat{N}$, A gen by A_1 & N simple. □

Earlier Examples

- Regular skew poly rings, i.e., $S = \mathbb{k}$-algebra on generators $z_1, \ldots, z_n$ where $z_j z_i = \mu_{ij} z_i z_j$ for all $i \neq j$ such that $\mu_{ij} \in \mathbb{k}^\times$ & $\mu_{ij}\mu_{ji} = 1$. In this case, $z_i = y_i$ from the theorem.
- ([CV1]) Regular graded Clifford algebras (with $\mathrm{char}(\mathbb{k}) \neq 2$).
- ([LV]) Homogenization of the universal enveloping algebra of any Lie algebra of dimension $n - 1 < \infty$.

More Examples

- ([CV1]) Regular graded skew Clifford algebras (with $\mathrm{char}(\mathbb{k}) \neq 2$).
- ([FRT], [GL], [LS], [L]) Coordinate ring of quantum $m \times \ell$ matrices, where $n = m\ell$.
- ([Lus], [DKP], [Y]) The subalgebras $\mathcal{U}^+[w]$ of $\mathcal{U}_q(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra, $\mathcal{U}_q(\mathfrak{g})$ is the quantized universal enveloping algebra of $\mathfrak{g}$, $q \in \mathbb{k}^\times$ is not a root of unity, & w is an element (of length n) of the associated Weyl group.

Conclusion

Definition (revisited)

Let A be as in the main theorem. If F is a normalizing sequence in A that consists of n homogeneous elements of positive degree, call $A/\langle F \rangle$ a **complete intersection** if the equivalent conditions I-IV hold.

Remarks

- All the above examples are Auslander-regular and Artin-Schelter regular. So, for those of us who view a regular algebra as being a non-commutative analogue of the poly ring, this is perhaps further support of that viewpoint.
- Since, in general, not all algebras have normalizing elements (or enough of them), other notions of complete intersection for non-commutative algebras are being investigated by researchers, such as the recent work in [KKZ] that uses a more homological approach than that used here.

References

[ATV1] M. Artin, J. Tate & M. Van den Bergh, Some Algebras Associated to Automorphisms of Elliptic Curves, *The Grothendieck Festschrift* **1**, 33-85, Eds. P. Cartier et al., Birkhäuser (Boston, 1990).

[ATV2] M. Artin, J. Tate and M. Van den Bergh, Modules over Regular Algebras of Dimension 3, *Invent. Math.* **106** (1991), 335-388.

[AZ] M. Artin and J. J. Zhang, Noncommutative Projective Schemes, *Adv. Math.* **109** No. 2 (1994), 228-287.

[CV1] T. Cassidy & M. Vancliff, Generalizations of Graded Clifford Algebras and of Complete Intersections, *J. Lond. Math. Soc.* **81** (2010), 91-112.

[CV2] T. Cassidy & M. Vancliff, Corrigendum to "Generalizations of Graded Clifford Algebras and of Complete Intersections" *J. London Math. Soc.*, to appear; www.uta.edu/math/vancliff/R/scliff-cor.pdf

[DKP] C. De Concini, V. Kac & C. Procesi, Some Quantum Analogues of Solvable Lie Groups, *Geometry and Analysis (Bombay 1992)*, 41-65, Tata Inst. Fund. Res. (Bombay, 1995).

[FRT] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtadzhyan, Quantization of Lie Groups and Lie Algebras, *Leningrad Math. J.* **1** No. 1 (1990), 193-225.

[GL] K. R. Goodearl and E. S. Letzter, Prime Factor Algebras of the Coordinate Ring of Quantum Matrices, *Proc. Amer. Math. Soc.* **121** No. 4 (1994), 1017-1025.

[KKZ] E. Kirkman, J. Kuzmanovich and J. J. Zhang, Noncommutative Complete Intersections, preprint (2013).

[LV] L. Le Bruyn and M. Van den Bergh, On Quantum Spaces of Lie Algebras, *Proc. Amer. Math. Soc.* **119** No. 2 (1993), 407-414.

[L] T. Levasseur, Some Properties of Non-commutative Regular Graded Rings, *Glasgow Math. J.* **34** (1992), 277-300.

[LS] T. Levasseur and J. T. Stafford, The Quantum Coordinate Ring of the Special Linear Group, *J. Pure & Appl. Algebra* **86** (1993), 181-186.

[Lus] G. Lusztig, *Introduction to Quantum Groups*, Progress in Math. **110**, Birkhäuser (Boston, 1993).

[NV] C. Năstăcescu and F. Van Oystaeyen, *Graded Ring Theory*, North-Holland (Amsterdam, 1982).

[SSW] L. W. Small, J. T. Stafford and R. B. Warfield, Affine Algebras of Gelfand-Kirillov Dimension One are PI, *Math. Proc. Cambridge Philos. Soc.* **97** (1985), 407-414.

[Y] M. Yakimov, A Proof of the Goodearl-Lenagan Polynormality Conjecture, *Int. Math. Res. Notices* **2013** No. 9, 2097-2132.