Classifying Quadratic Quantum Planes using Graded Skew Clifford Algebras

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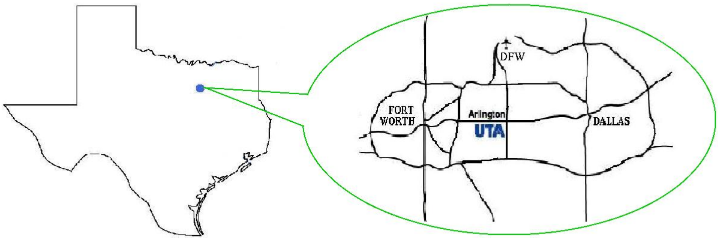
with Manizheh Nafari & Jun Zhang



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Henceforth, $\mathbb{k}$ = algebraically closed field.

μ-symmetric Matrices

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Let $\mu = (\mu_{ij}) \in M(n, \Bbbk)$ be such that $\mu_{ij}\mu_{ji} = 1$ for all i, j such that $i \neq j$.

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Assumption

For the rest of the talk, assume $\mu_{ii} = 1$ for all i.

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Example

Skew polynomial rings on generators $x_1, \ldots, x_n$ with relations $x_i x_j = -\mu_{ij} x_j x_i$, for all $i \neq j$, are GSCAs.

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We call any (nonzero) element of S_2 a quadratic form, and define the quadric, $\mathcal{V}(q)$, determined by any quadratic form q to be the set of points in $\mathbb{P}(S_1^*) \times \mathbb{P}(S_1^*)$ on which q and the defining relations of S vanish.

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Theorem ([Cassidy, Vancliff])

A GSCA $A = A(\mu, M_1, \ldots, M_n)$ *is a quadratic, Auslander-regular algebra of global dimension* n *that satisfies the Cohen-Macaulay property with Hilbert series* $1/(1-t)^n$ *iff*

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Note: our work attempts to classify all quadratic regular algebras D of global dimension 3; not only the generic ones.

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If X is a nodal cubic curve, then $D = \Bbbk[x_1, x_2, x_3]$ *with defining relations:*

$$\lambda x_1 x_2 = x_2 x_1, \qquad \lambda x_2 x_3 = x_3 x_2 - x_1^2, \qquad \lambda x_3 x_1 = x_1 x_3 - x_2^2,$$

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Suppose char($\Bbbk$) $\neq 2$.

- *If $\lambda^3 \notin \{0, 1\}$, then D is an Ore extn of a regular GSCA of gldim 2;*
- *if $\lambda^3 = -1$, then D is a GSCA.*

Theorem

X = cuspidal cubic curve in $\mathbb{P}^2$ iff char($\mathbb{k}$) $\neq 3$ & $D = \mathbb{k}[x_1, x_2, x_3]$ with def rels:

$$x_1x_2 = x_2x_1 + x_1^2, \quad x_3x_1 = x_1x_3 + x_1^2 + 3x_2^2, \quad x_3x_2 = x_2x_3 - 3x_2^2 - 2x_1x_3 - 2x_1x_2.$$

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It remains to consider X = elliptic curve in $\mathbb{P}^2$.

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(Moreover, any such algebra is regular, even if char$(\Bbbk) = 3$.)

If char$(\Bbbk) \neq 2$ & $X =$ cuspidal cubic curve, then D is an Ore extn of a regular GSCA of gldim 2.

It remains to consider $X =$ elliptic curve in $\mathbb{P}^2$.

In [AS, ATV1], such algebras are classified into types A, B, E, H,

where some members of each type might not have an elliptic curve as their point scheme, but a generic member does.

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In (iii), $a^3 \neq b^3 \neq c^3 \neq a^3$ is still open.

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- Can quadratic regular algebras of gldim 4 be classified using GSCAs?